Lecture 6: asymptotes

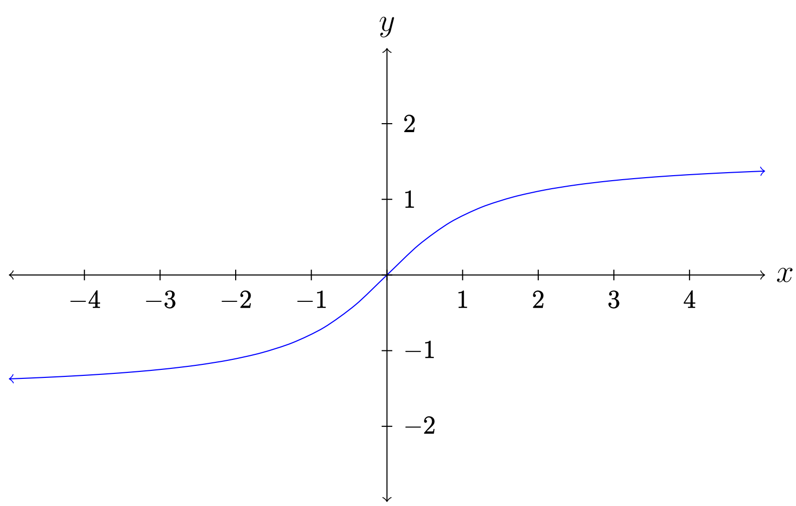
Calculus I, section 10
September 22, 2022

Last time we talked about continuous functions and various kinds of discontinuities. Today we'll say more about certain kinds of discontinuities, together with generally trying to understand the behavior of functions "at infinity." The general principle will be that often, a function may be very complicated for small numbers, or even large numbers, but once we get large enough—in any direction—we expect things to be simpler.

The most straightforward way of talking about the behavior of functions at infinity is in terms of infinite limits, as we've seen before: $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$. These limits might or might not converge.

If they don't converge, we (so far) can't say much about the function "at infinity"—it does some mysterious thing. If they do converge, though, then whatever the function is for finite numbers, once x is very large we can get a good approximation to $f(x)$ using this limit: if $\lim_{x\to\infty} f(x) = L$, then for x "near infinity," i.e. very large, $f(x)$ is well approximated by L. We can see this on a graph: as we go out to infinity, the graphs of $y = f(x)$ and $y = L$ approach each other. In this case we say that $y = f(x)$ has a horizontal asymptote at $y = L$.

The same thing is true as $x \to -\infty$. These limits may be the same, or may be different, so we can have up to two horizontal asymptotes. A classic example is the arctangent function $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, the inverse of the restriction of the tangent function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. Its graph looks like this:



As $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$, and as $x \to -\infty$ $\tan^{-1}(x) \to -\frac{\pi}{2}$, so $\tan^{-1}(x)$ has two horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$. On the other hand $f(x) = \frac{1}{x}$ has only one horizontal asymptote: $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$, so the only horizontal asymptote is at $y = 0$.

For rational functions specifically, there's a useful trick for infinite limits you may or may not be familiar with. A rational function is of the form $\frac{f(x)}{g(x)}$ for f and g polynomials. If f has a higher degree (i.e. the highest power in its expansion) than g, then as $x \to \infty$ the numerator will get much bigger than the denominator, and so the limit does not exist; the same thing is true as $x \to -\infty$. If g has a higher degree than f, then the denominator will get much bigger than the numerator and so the limit is 0. If they have the same degree, then only the largest terms of each will end up mattering because for x very large, e.g. x^4 is much bigger than x^3, so we can neglect the lower-order terms: for example, we can immediately conclude that

$$\lim_{x\to-\infty} \frac{2x^3 - 4x^2 + x + 9}{3x^3 - x - 1} = \lim_{x\to-\infty} \frac{2x^3}{3x^3} = \frac{2}{3}.$$

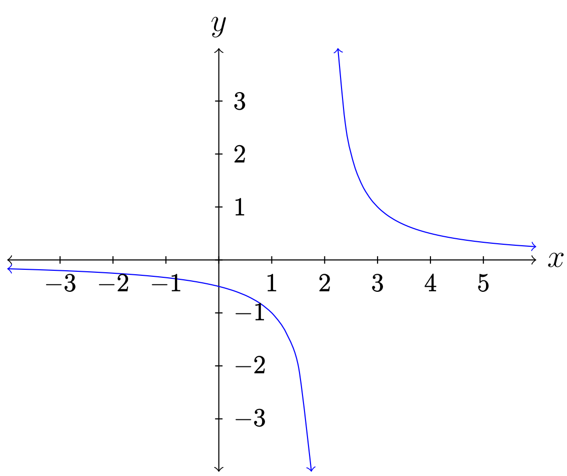
This makes finding horizontal limits of rational functions much easier.

As the name suggests, in addition to horizontal asymptotes we can also have vertical asymptotes. This is a different kind of behavior at infinity: instead of looking at what happens "near $x = \infty$," we look for points where y goes to infinity (but x is finite, just as we wanted y to be finite for horizontal asymptotes). These are given by certain kinds of discontinuities we saw last time. Specifically, we are looking for essential discontinuities (since something is going to infinity, at least one of the one-sided limits must not exist), and specifically those where at least one of the one-sided limits goes to infinity, rather than failing to exist in a different way. These are sometimes called poles, as we mentioned the other day.

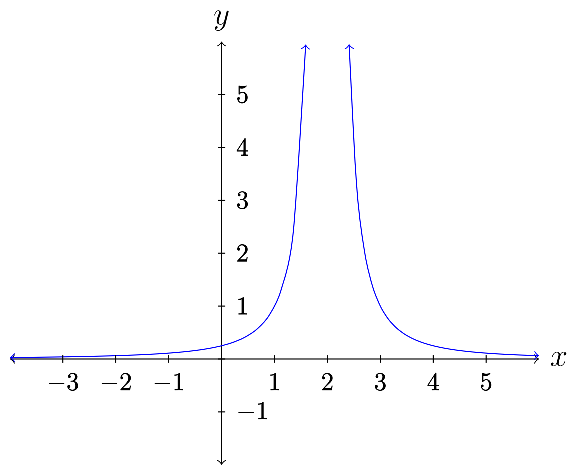
For example, $f(x) = \frac{x}{x^2-2x}$ has discontinuities at $x = 0$ and $x = 2$, since that's where the denominator vanishes. At $x = 0$, we have a removable discontinuity: we can cancel factors of x to see that away from 0, this is the same thing as $\frac{1}{x-2}$, which is defined (and continuous) at 0 (with a value of $-\frac{1}{2}$), so the limit exists. At $x = 2$, we have an essential discontinuity, and it does in fact blow up to infinity: in the limit we get something of the form $\frac{2}{0}$, which must go to infinity somewhere. Therefore the only vertical asymptote is at $x = 2$.

While we're at it, what are the horizontal asymptotes of this function, if any?

There's a little more to be said about vertical asymptotes. For example, the function above, which we can simplify to $\frac{1}{x-2}$ away from 0, looks like this:



This is in some sense qualitatively different from the graph of $y = \frac{1}{(x-2)^2}$, which looks like this:



Most noticeably, in this second example both one-sided limits go to $+\infty$, instead of going in different directions. Additionally, if you look closely you might guess that they're going to infinity "faster" in some sense than the previous example.

This is an instance of the phenomenon of poles having *order*: the idea is that often, whatever kind of pole we're looking at (even for non-rational functions) if it's at a point a, the function near a should be roughly given by $\frac{1}{(x-a)^n}$ for some positive integer n, called the order of the pole. The higher n is, the "faster" the function will blow up as we go closer to a; if n is odd, then we'll see the behavior of going to infinity in two different directions, and if it's even we'll see both directions going to $+\infty$.

This is really more of a real analysis topic and I won't ask you about it. It is worth pointing out that there are also kinds of singularities (even which go to infinity) which do not fit this pattern: for example, the singularity of $\log_b(x)$ at $x = 0$ is not of this form (it's called a logarithmic singularity instead). Another classic example is something like the function $f(x) = 3^{-\frac{1}{x^2}}$. What can we say about this? Where might it have asymptotes or discontinuities?

First, we can look at horizontal asymptotes: $\lim_{x\to\infty} 3^{1/x} = 3^0 = 1$, by the composition limit law, and the same thing is true as we go to $-\infty$. As far as discontinuities, 3^x is continuous everywhere, and $\frac{1}{x}$ is continuous everywhere except $x = 0$, so the only possible discontinuity is at $x = 0$.

At $x = 0$, what happens? Well, first let's take the limit from above: $\lim_{x\to 0^+} 3^{1/x} = \lim_{y\to+\infty} 3^y$ goes to infinity, using the composition limit law. This more or less is what we expect.

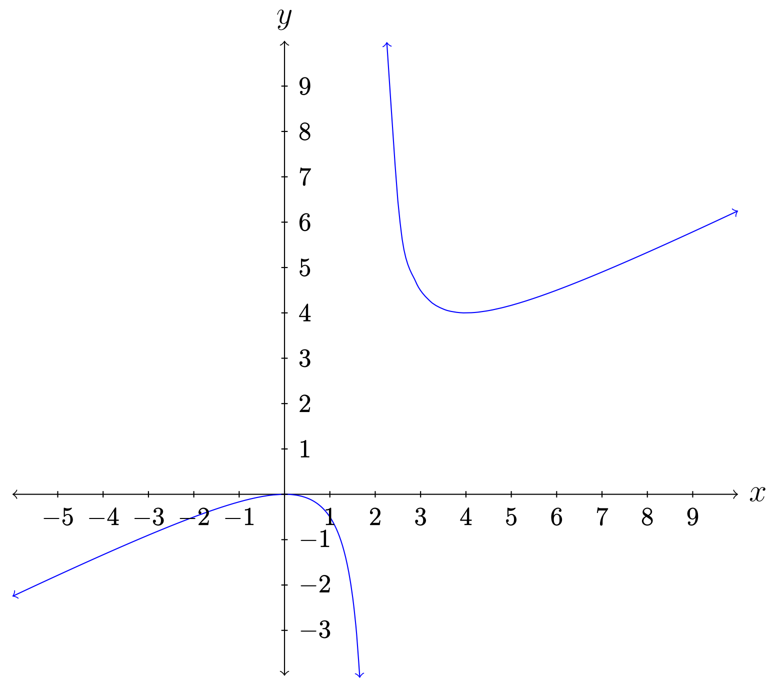
On the other hand, $\lim_{x\to 0^-} 3^{1/x} = \lim_{y\to-\infty} 3^y = 0$, by the same argument. This means that we have a singularity where one side goes to infinity and the other side is well-behaved, and actually goes to zero!

This is very strange behavior, and it turns out that it in a certain sense corresponds to a pole of infinite order. It's sometimes called an essential singularity, which is confusing terminology since it is only a special kind of essential discontinuity; you won't have to use this terminology at least in this class so hopefully it won't be too confusing, but it's good to see wilder types of singularities. In particular here even though one side of the limit at

$x = 0$ is defined and goes to 0, we still have a vertical asymptote at $x = 0$.

Now, we can also have asymptotes that are neither horizontal nor vertical, but these are a little more subtle. For horizontal or vertical asymptotes, visually we can think about looking at lines on the graph; since these represent places where $y = f(x)$ is close to a constant graph $x = a$ or $y = a$, they are places where we can give a simpler description even though the function itself may be very complicated. We could also think of other lines on a graph which a function might get close to, which could be at angles other than horizontal or vertical; in other words, diagonal asymptotes.

Consider for example $f(x) = \frac{x^2}{2x-4}$.



This has a vertical asymptote at $x = 2$, and we can check that there are no horizontal asymptotes since the infinite limits don't exist: the numerator has larger degree than the denominator.

However, if we try to pull the same trick and only look at the leading terms, we see that if x is large then

$$f(x) \approx \frac{x^2}{2x} = \frac{1}{2}x,$$

suggesting that for x (or $-x$) large our function is closely approximated by a linear function, just not a constant one. Any linear function is of the form $y = mx + b$; since $f(x) \approx \frac{1}{2}x$, we get that the slope m should be $\frac{1}{2}$. To figure out what the constant term might be, look at $f(x) - \frac{1}{2}x$. We can expand this as

$$\frac{x^2}{2x-4} - \frac{x}{2} = \frac{x^2}{2x-4} - \frac{x(x-2)}{2x-4} = \frac{2x}{2x-4} = 1 + \frac{4}{2x-4},$$

which for $|x|$ very large approaches 1. Therefore $f(x) \approx \frac{1}{2}x + 1$ and so there is a diagonal asymptote at $y = \frac{1}{2}x + 1$.

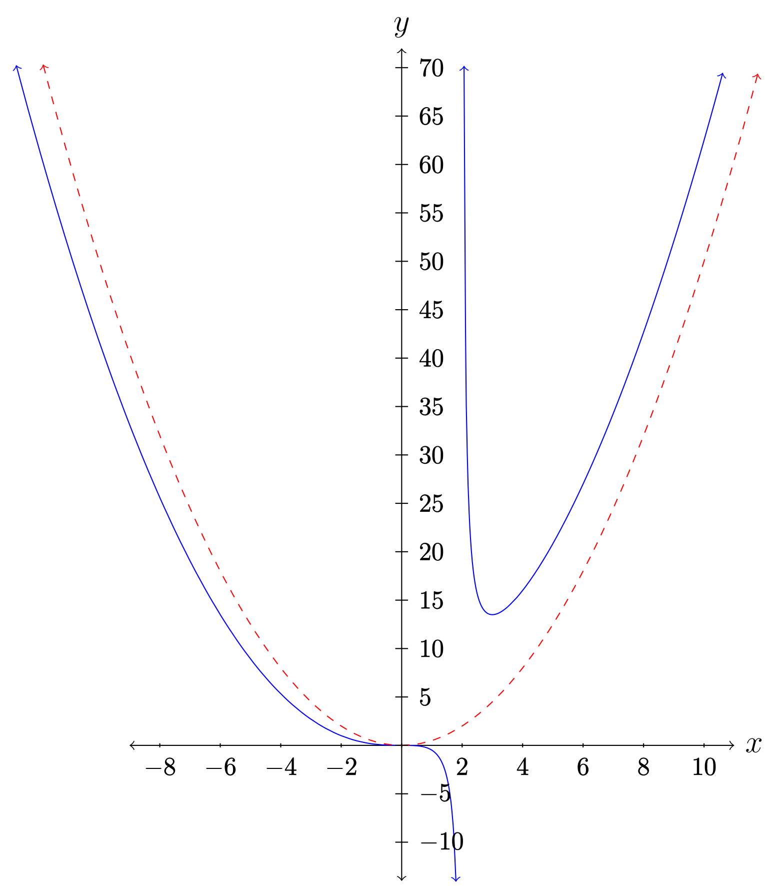
Warning: it might be tempting to try to evaluate the constant term by looking at $f(0)$, which in this case would be 0. But $y = mx + b$ is only a good approximation for $f(x)$ if x (or $-x$) is very large, and at $x = 0$ there's no reason for $f(0)$ and b to be closely related.

Just like for horizontal asymptotes, we can in principle have up to two, one as $x \to +\infty$ and one as $x \to -\infty$; often, though not always, these will be the same, if they exist at all. (Giving an example where there are two different diagonal asymptotes is part of one of your homework problems for this week.)

These are the kinds of features "at infinity" which are easy to notice on a graph: lines stand out. Indeed one way of thinking about asymptotes is as connecting these visual features to what's mathematically going on in terms of infinite limits. Mathematically speaking, though, there are many other kinds of behavior "at infinity." For example, suppose instead of the example above we had $f(x) = \frac{x^3}{2x-4}$. Then for $|x|$ very large,

$$f(x) \approx \frac{x^3}{2x} = \frac{1}{2}x^2.$$

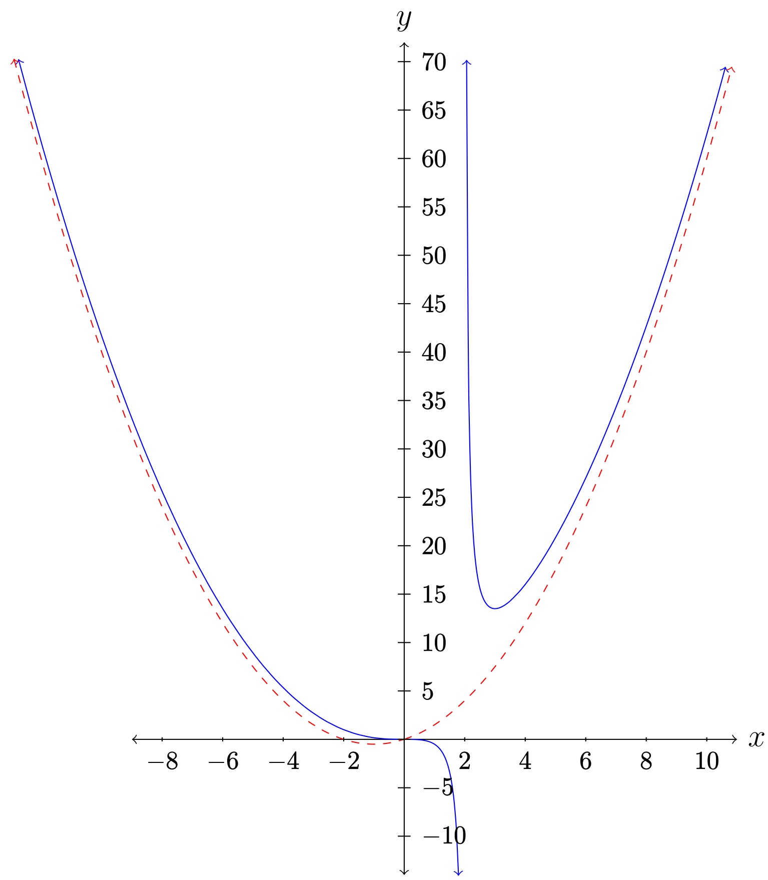
This is not linear, and we won't be able to see a linear asymptote, diagonal or otherwise, on the graph; nevertheless it's still useful to be able to approximate complicated functions by simpler ones, and we can still do this here:



This is a pretty decent approximation, but we can see it's not perfect, analogous to how we needed to add a constant term before. Indeed, subtracting we get

$$f(x) - \frac{1}{2}x^2 = \frac{x^3}{2x-4} - \frac{1}{2}x^2 = \frac{x^3}{2x-4} - \frac{x^2(x-2)}{2x-4} = \frac{x^2}{x-2} \approx \frac{x^2}{x} = x,$$

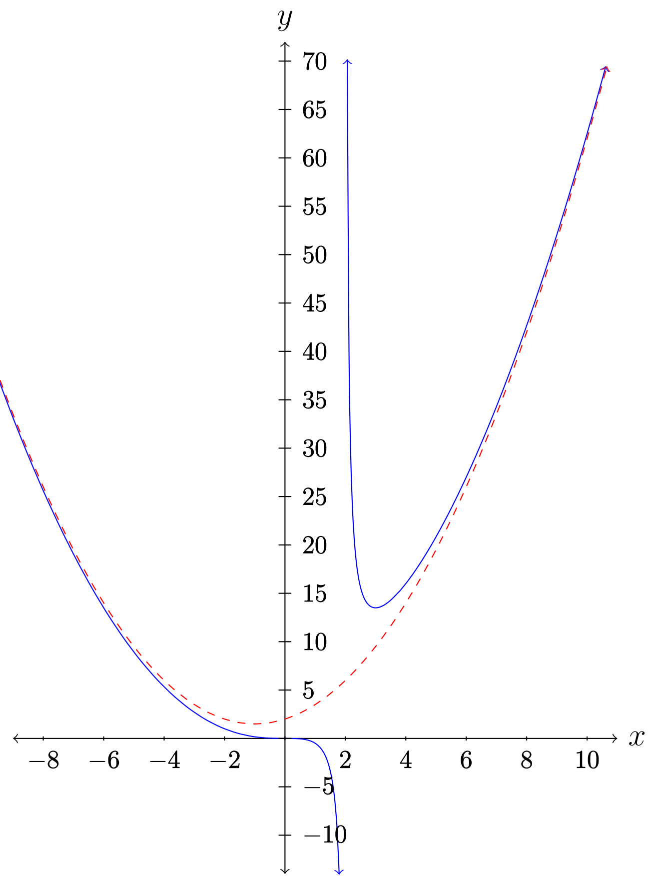
so a better approximation is $f(x) \approx \frac{1}{2}x^2 + x$.



This is better, but still not perfect; we can improve it by adding one more term (since we're now looking for a quadratic approximation), which we find by subtracting again:

$$f(x) - \left(\frac{1}{2}x^2 + x\right) = \frac{x^3}{2x-4} - \frac{1}{2}x^2 - x = \frac{x^3}{2x-4} - \frac{(x^2+2x)(x-2)}{2x-4} = \frac{4x}{2x-4} \approx 2$$

and so our approximation is now $f(x) \approx \frac{1}{2}x^2 + x + 2$.



Once $|x|$ is large this is quite good. (To do even better, we could continue the above process with polynomial long division, which is essentially what we're doing here: this would give us

$$f(x) = \frac{1}{2}x^2 + x + 2 + \frac{8}{2x-4} \approx \frac{1}{2}x^2 + x + 2 + \frac{4}{x},$$

and we could keep going indefinitely.) This is the same idea as horizontal or diagonal asymptotes: if horizontal asymptotes are the constant approximation to our function at infinity, if one exists, and diagonal asymptotes are the linear approximations, then this is a quadratic approximation, and one can also do higher-order versions.