Lecture 9: chain rule

Calculus I, section 10
October 6, 2022

Before the midterm, we introduced derivatives and saw a number of ways to compute them. We can directly use the limit definition, or in some cases we have rules to compute them, such as the power rule; if our function is formed as a sum, difference, product, or quotient of two functions whose derivatives we know, we can compute the derivative using linearity or the product or quotient rules.

Let's take some time to review these, since it's been a little and we didn't have a lot of examples before. Consider a function like $f(x) = x^2 \sin(x) + \tan(x)$. By linearity,

$$f'(x) = \frac{d}{dx} x^2 \sin(x) + \frac{d}{dx} \tan(x),$$

so let's take the terms one at a time. For the first term, we have two factors, each of which we know how to differentiate, so we apply the product rule:

$$\begin{aligned} \frac{d}{dx} x^2 \sin(x) &= \left(\frac{d}{dx} x^2\right) \cdot \sin(x) + x^2 \cdot \left(\frac{d}{dx} \sin(x)\right) \\ &= 2x \sin(x) + x^2 \cos(x). \end{aligned}$$

For the second term, we could remember

$$\frac{d}{dx} \tan(x) = \sec(x)^2.$$

Personally I can never remember any of the derivatives of trigonometric functions except sine and cosine, so let's use those plus the quotient rule:

$$\begin{aligned} \frac{d}{dx} \tan(x) &= \frac{d}{dx} \frac{\sin(x)}{\cos(x)} \\ &= \frac{\sin'(x) \cos(x) - \sin(x) \cos'(x)}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} \\ &= \frac{1}{\cos(x)^2} \\ &= \sec(x)^2. \end{aligned}$$

Therefore in total

$$f'(x) = 2x \sin(x) + x^2 \cos(x) + \sec(x)^2.$$

We're slowly progressing towards our goal of being able to differentiate any function we can write down. However, we're not there yet: we still don't know what to do with a function like $f(x) = \sin(x^2)$. It's a trigonometric function, sort of, but not one of the ones we know, and we can't make it into a product or quotient, so none of our tricks apply.

What should we do? Well, the only thing to do is go back to the definition:

$$f'(x) = \lim_{h \to 0} \frac{\sin((x+h)^2) - \sin(x^2)}{h}.$$

We could expand out $(x+h)^2 = x^2 + 2hx + h^2$ and then apply the angle addition formulas again, multiple times; maybe at the end of this we'll get something reasonable.

This is going to be very painful, and it's easy to break this sort of method: what if instead we had something like $\sin(2^x)$? Instead, let's pause to think about what we're doing, heuristically.

The derivative, as the notation $\frac{df}{dx}$ suggests, is supposed to be the ratio of the change in f to the change in x, as this change goes to 0. The situation that we're in here is that instead of a reasonably simple function f, into which we can plug x, we have a reasonably simple function g, into which we plug another reasonably simple function h: $f(x) = g(h(x))$. If we write $y = h(x)$, so $f(x) = g(y)$, then $\frac{df}{dx}$ could be thought of as the ratio of the change in g to the change in y, corrected by the change in y relative to the change in x:

$$f'(x) = \frac{dg}{dx} = \frac{dg}{dy} \cdot \frac{dy}{dx} = g'(y)\frac{dy}{dx} = g'(h(x))h'(x).$$

This is purely heuristic: $\frac{dg}{dy}$ and so on are not literally fractions and so we can't literally cancel in this way; nevertheless this is a good way of thinking of it, and it'll turn out to be true.

In our example, we can take $g(x) = \sin(x)$ and $h(x) = x^2$. First, we differentiate the outer function: $\sin'(x) = \cos(x)$. We want to evaluate this not at x, but at $y = h(x)$, so $\cos(x^2)$. Finally, we correct by the derivative of the inner term to get $2x\cos(x^2)$.

Warning: it is very easy to get confused with all these steps, and end up with something like $2x\cos(x)$ (evaluating at x, instead of $h(x)$) or $\cos(x^2)$ (forgetting the correcting factor $h'(x)$). We need all the parts for this to be true!

But why is it true? We have to go back again to the limit definition of the derivative. Let's change notation a little and try to compute

$$\frac{d}{dx} f(g(x)).$$

The key idea is to use the derivative as a linear approximation: $g(x+h) \approx g(x) + hg'(x)$. Thus

$$\begin{aligned}
\frac{d}{dx} f(g(x)) &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(g(x) + hg'(x)) - f(g(x))}{h} \\
&= \lim_{j \to 0} \frac{f(g(x) + j) - f(g(x))}{j/g'(x)}
\end{aligned}$$

where $j = hg'(x)$, so that (so long as $g'(x)$ is nonzero) h going to zero is the same thing as j going to zero. Then this is

$$\lim_{h\to 0} \frac{f(g(x)+j) - f(g(x))}{h} \cdot g'(x) = f'(g(x))g'(x).$$

Thus we've proven the *chain rule*:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

If you were watching carefully, you might have noticed that we assumed $g'(x) \neq 0$ to do our limit switch, so you might be worried this isn't true if $g'(x) = 0$. However, if $g'(x) = 0$, then the second row above becomes

$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x)) - f(g(x))}{h} = 0,$$

which agrees with the chain rule if $g'(x) = 0$, so the formula always works.

Let's look at some more examples. What about something like $f(x) = \tan(x)^4 + 3\tan(x)^2 + \tan(x) - 3$? We could calculate the derivative using our knowledge of the derivative of $\tan(x)$ together with linearity and the product rule, but it would be difficult. Much easier is to set $g(x) = x^4 + 3x^2 + x - 3$ and $h(x) = \tan(x)$, so that $f(x) = g(h(x))$. Then

$$f'(x) = g'(h(x))h'(x) = (4\tan(x)^3 + 6\tan(x) + 1)\sec(x)^2.$$

A trickier example is our old enemy

$$f(x) = \sin\left(\frac{1}{x}\right).$$

If $g(x) = \sin(x)$ and $h(x) = \frac{1}{x}$, then

$$f'(x) = \cos\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2} = -\frac{\cos(\frac{1}{x})}{x^2},$$

which (unsurprisingly) fails badly to exist at $x = 0$.

We'll sometimes encounter polynomials of the form $f(x) = (2x+3)^4 - x^2$ or similar, where we could expand everything out and apply linearity or the power rule, but not without great unpleasantness. Alternatively, we can apply the chain rule: let $g(x) = x^4$ and $h(x) = 2x+3$. Then $f(x)$ is not quite $g(h(x))$, but $g(h(x)) + x^2$. We can apply linearity and then the chain and power rules: $f'(x) = g'(h(x))h'(x) + 2x = 4(2x+3)^3 \cdot 2 + 2x = 8(2x+3)^3 + 2x$.

Another interesting example is $f(x) = \sin(2x)$. This is a straightforward application of the chain rule: the derivative of the inside is 2, the derivative of the outside is $\cos(y)$, so the whole thing is $f'(x) = 2\cos(2x)$.

On the other hand, we could also use trigonometry and the *product* rule: you might recall that the double angle formula states that $\sin(2x) = 2\sin(x)\cos(x)$. Therefore its derivative is $2(\cos(x)^2 - \sin(x)^2)$. Comparing these two formulas gives the double angle formula for cosine:

$$\cos(2x) = \cos(x)^2 - \sin(x)^2.$$

We can also look back at some rules we've seen before. For example, we saw from the quotient rule that

$$\frac{d}{dx}\frac{1}{f(x)} = -\frac{f'(x)}{f(x)^2}$$

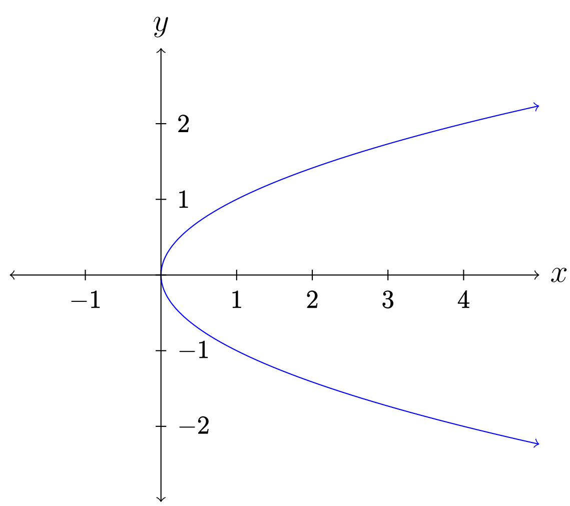
(the reciprocal rule). We can also see it as the consequence of the chain rule: if $g(x) = \frac{1}{x}$, then

$$\frac{d}{dx}\frac{1}{f(x)} = \frac{d}{dx}g(f(x)) = g'(f(x))f'(x) = -\frac{1}{f(x)^2}\cdot f'(x) = -\frac{f'(x)}{f(x)^2}.$$

(In fact, we could then derive the quotient rule again from this reciprocal rule together with the product rule.)

Another kind of application is called *implicit differentiation*. Sometimes we have some kind of relationship between x and y that cannot be expressed neatly as a function (or which is more convenient not to), whether because it is not a function (such as $y^2 = x$) or because solving analytically is difficult or impossible (such as $y^5 + x + y = 1$). We might still want to know $\frac{dy}{dx}$, i.e. the rate of change of y with respect to x at a given point; this could look like finding the slope of a graph, even if it's not the graph of a function, or more abstract things.

The idea here is that, analogous to algebraic equations, we *differentiate both sides* with respect to x. For example, take the example $y^2 = x$, which we could also write as $y = \pm\sqrt{x}$ (so this is not a function, but still something we can describe explicitly).



Then differentiating both sides gives

$$\frac{d}{dx}y^2 = \frac{d}{dx}x.$$

Since y depends on x in some way, we apply the chain rule on the left:

$$\frac{d}{dx}y^2 = \left(\frac{d}{dy}y^2\right)\cdot\frac{d}{dx}y = 2y\cdot\frac{dy}{dx} = 2yy',$$

and the right-hand side is simply

$$\frac{d}{dx}x = 1.$$

Therefore we have

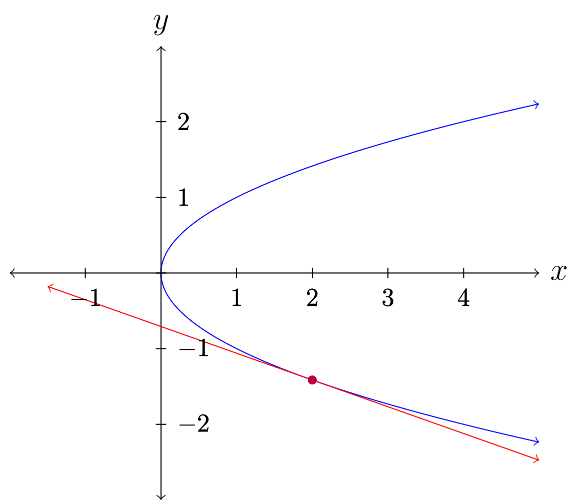
$$2yy' = 1,$$

and so

$$y' = \frac{1}{2y}.$$

Thus if we want to find the slope of a particular point, say $x = 2$, $y = -\sqrt{2}$, then we plug in these values to get

$$\left.\frac{dy}{dx}\right|_{(2,-\sqrt{2})} = \frac{1}{2 \cdot (-\sqrt{2})} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}.$$



This is the idea of implicit differentiation: even if we don't know what y is as a function of x (or even if globally it isn't a function of x), we treat it as if it's one and then solve for y' at the end, instead of solving for y at the beginning (which may not be possible).

Notice, by the way, that in the previous example if we choose the positive branch $y \geq 0$, so that $y = \sqrt{x}$, then the implicit differentiation spits out

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}},$$

which is a formula we've seen before using the power rule for $n = \frac{1}{2}$. But here we only had to use the power rule for positive integers (namely 2), as well as the chain rule.

In fact, we can do this trick in general for fractional powers. Let's start with things of the form $y = x^{1/n} = \sqrt[n]{x}$. Then in particular we have $y^n = x$, so differentiating both sides and applying the chain rule we get

$$ny^{n-1}y' = 1$$

and so

$$y' = \frac{d}{dx}x^{1/n} = \frac{1}{ny^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1},$$

exactly as the power rule predicts.

Now we can use the chain rule to get all fractional exponents: $x^{m/n} = (x^{1/n})^m$, so

$$\frac{d}{dx}x^{m/n} = m(x^{1/n})^{m-1} \cdot \frac{d}{dx}x^{1/n} = mx^{\frac{m-1}{n}} \cdot \frac{1}{n}x^{\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m}{n}-1}$$

as desired.

We can also do something quite powerful with implicit differentiation which is useful even just for regular functions: we can differentiate inverse functions. Recall that if $f(x)$ is a function $\mathbb{R} \to \mathbb{R}$, its inverse function $f^{-1}(x)$ (not to be confused with $f(x)^{-1} = \frac{1}{f(x)}$) is a function such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ for all x. Such a function may or may not exist, or may exist only after restricting the domain and/or codomain.

Suppose that f does have an inverse function f^{-1}, and let's say that we understand the derivative of f and want to understand the derivative of f^{-1}. Then we can do essentially the same operation as above. If $y = f^{-1}(x)$, then $x = f(y)$ by the definition of inverse functions; so instead of differentiating $y = f^{-1}(x)$ directly, we instead differentiate $x = f(y)$, and then we'll try and solve at the end. We get

$$\frac{d}{dx}x = \frac{d}{dx}f(y).$$

As above, the left-hand side is just 1; and on the right-hand side we apply the chain rule to get $f'(y)y'$. Therefore

$$y' = \frac{d}{dx}f^{-1}(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

A simple example is one we've already seen several times, now from yet another angle: $\sqrt{x}$ is the inverse function of x^2 (after restricting the domain and codomain) and the derivative of x^2 is $2x$, so

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

A more complicated example uses trigonometric functions. Consider

$$f(x) = \tan^{-1}(x),$$

the inverse function of the tangent. We know from above that $\tan'(x) = \sec(x)^2$, so the formula above gives

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec(\tan^{-1}(x))^2}.$$

This is still a pretty messy formula, but we can simplify it, using a similar technique to one of your homework problems. If we start with the identity $\sin(\theta)^2 + \cos(\theta)^2 = 1$ and divide everything by $\cos(\theta)^2$, we get $\tan(\theta)^2 + 1 = \sec(\theta)^2$, so if $\theta = \tan^{-1}(x)$ then $\sec(\theta)^2 = \tan(\tan^{-1}(x))^2 + 1 = x^2 + 1$. Therefore we conclude that

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2 + 1},$$

which is very far from obvious!