Lecture 10: examples and differentiability

Calculus I, section 10
October 5, 2023

Before moving to differentiability, let's work through some more examples of the chain rule and practice combining different rules.

Consider $f(x) = \frac{1}{(1-x)^5}$. If $g(x) = \frac{1}{x^5}$ and $h(x) = 1 - x$, then $f(x) = g(h(x))$, so $f'(x) = g'(h(x))h'(x) = -\frac{5}{(1-x)^6} \cdot (-1) = \frac{5}{(1-x)^6}$.

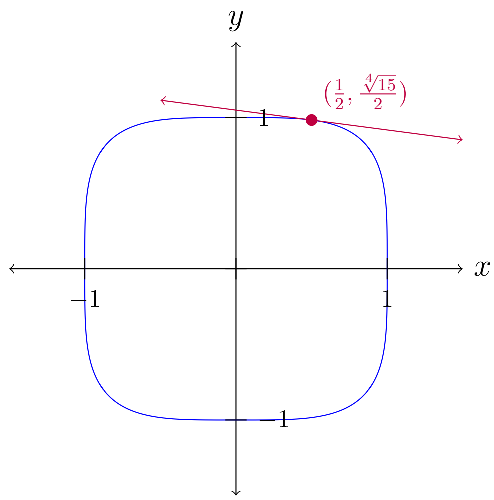
Another, more complicated example is something like $\frac{d}{dx} x \sin(x^3)$. We want to apply the product rule here, viewing this as the product of x and $\sin(x^3)$; but to do so we need to know the derivatives of both terms, and while the first is easy (the derivative is just 1) the second is harder. We apply the chain rule: $\frac{d}{dx} \sin(x^3) = \cos(x^3) \cdot 3x^2$, so the overall derivative is

$$\frac{d}{dx} x \sin(x^3) = \sin(x^3) + x \cdot \cos(x^3 \cdot 3x^2 = \sin(x^3) + 3x^3 \cos(x^3).$$

We can also apply our knowledge of derivatives of inverse functions for a few more notable cases. We saw last time that $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2+1}$; we can also compute $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}$, which by a similar argument to last time is $\frac{1}{\sqrt{1-x^2}}$. Similarly $\frac{d}{dx} \cos^{-1}(x) = \frac{1}{-\sin(\cos^{-1}(x))} = -\frac{1}{\sqrt{1-x^2}}$.

This is interesting in that the derivatives of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are almost the same, namely negatives of each other. In particular this implies that $\frac{d}{dx}(\sin^{-1}(x) + \cos^{-1}(x)) = 0$ (by linearity). The only functions we've seen before whose derivative is everywhere zero are constant functions, and in fact it turns out to be true that those are the only such functions, so it follows that $\sin^{-1}(x) + \cos^{-1}(x)$ is some constant c, independent of x. Plugging in $x = 0$, we get that $c = \sin^{-1}(0) + \cos^{-1}(0)$, which for our standard choice of domain and codomain for these inverse functions is $0 + \frac{\pi}{2} = \frac{\pi}{2}$, i.e. $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ for every x. In other words, if θ is an angle such that $\sin(\theta) = x$, i.e. $\theta = \sin^{-1}(x)$, then $\cos^{-1}(x) = \frac{\pi}{2} - \theta$, i.e. $\cos(\frac{\pi}{2} - \theta) = x = \sin(\theta)$. This recovers the relation $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$, which we know from trigonometry.

We recall also the idea of implicit differentiation: say x and y are related by $x^4 + y^4 = 1$, which looks like a slightly squared-off circle.



To compute the slope at a given point $(\frac{1}{2}, \frac{\sqrt[4]{15}}{2})$, we use implicit differentiation: $\frac{d}{dx}x^4 + \frac{d}{dx}y^4 = 4x^3 + 4y^3\frac{dy}{dx} = \frac{d}{dx}1 = 0$, so $\frac{dy}{dx} = -\frac{x^3}{y^3}$, which at our point is $-\frac{1/2^3}{15^{3/4}/2^3} = -15^{-3/4} \approx -0.1312$.

We now know how to find formulas for the derivatives of a wide variety of functions; but the derivative is defined by a limit, and while we spent a lot of time worrying about whether various limits exist we haven't really talked about when the derivative exists.

Certainly it does not always exist. For example, if $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$ and so $f'(0)$ does not exist. This is not surprising, as $f(0)$ doesn't exist either.

Similarly, consider the function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$

For $x < 0$ or $x > 0$, this is a constant function near x and so the derivative is just 0; but at $x = 0$, $f(x)$ has a jump discontinuity and so we don't expect the derivative to exist, since even if x changes only a very small amount $f(x)$ can still change by a large amount. Indeed,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1 - 1}{x} = 0,$$

but

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-1 - 1}{x} \to +\infty.$$

Therefore $f'(x)$ does not exist at $x = 0$.

This suggests a criterion: in order for $f(x)$ to be differentiable at a point a, it must at least be continuous at a. This is true: if $f(a)$ doesn't exist, then the derivative

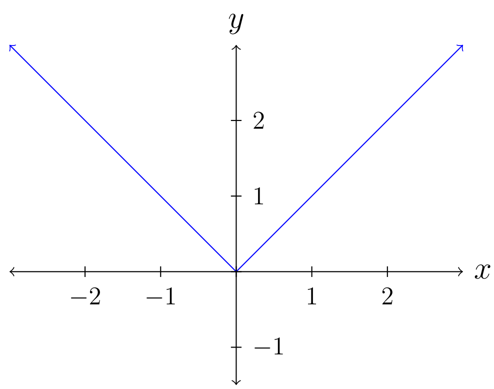
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

can't exist either, and if it exists but is different from $\lim_{x \to a} f(x)$ (or this limit doesn't exist) then the numerator of the limit defining the derivative either won't go to zero as $x \to a$ or won't exist at all in the limit; in either case we can't hope to have the limit converge.

Okay, so differentiable functions are continuous. Is the converse true, i.e. if $f(x)$ is continuous at a, does that imply that it's differentiable at a as well?

No! The standard counterexample is

$$f(x) = |x|.$$



At the point $x = 0$, we have

$$\lim_{x\to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0^+} \frac{x}{x} = 1,$$

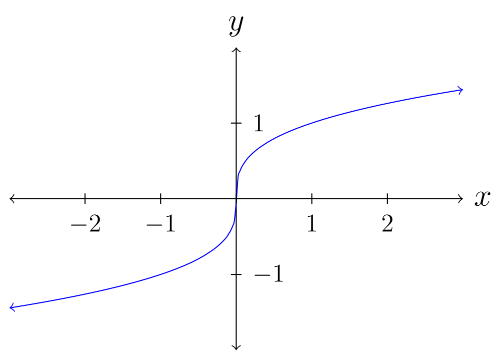
but

$$\lim_{x\to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$$

This conforms to our intuitions about derivatives and tangent lines: the tangent line to the graph from the right has slope 1, but from the left has slope -1. The problem here is the "pointiness" of $f(x)$ at $x = 0$: there isn't a single tangent line, and so the derivative doesn't make sense.

Another example is

$$f(x) = x^{1/3} = \sqrt[3]{x}.$$



This is continuous for all real numbers, with no obvious pointiness. But if you look closely near $x = 0$, you might guess that the slope is getting almost vertical; and indeed this is what happens. We have

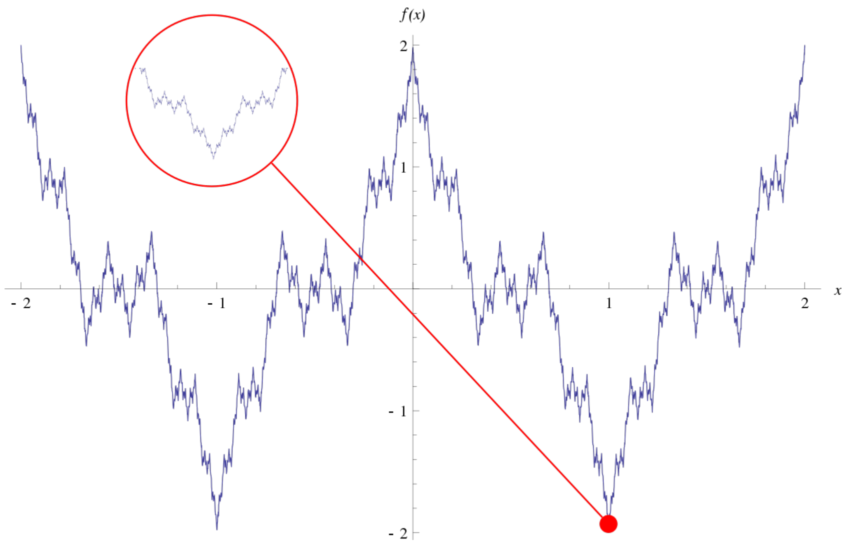
$$f'(x) = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}},$$

which is undefined at $x = 0$.

So what we've seen is: all differentiable functions are continuous, but not all continuous functions are differentiable. How can we check if a function is differentiable at a given point?

First, we can check if it's defined and continuous at that point: if not, it can't be differentiable either. Next, we can check directly to see if the limit exists, like we did for $|x|$, or we can use a formula for the derivative (if we have one) and then check if it makes sense at that value, like we did for $\frac{1}{3x^{2/3}}$. You'll get a little more practice on this on the homework.

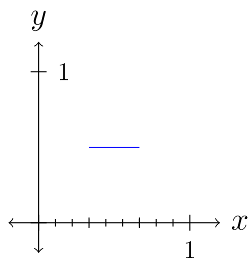
Finally, let's just look at some stranger examples of functions, to get a little better intuition for differentiability. A famous example is the Weierstrass function: it was generally believed that for an everywhere-continuous function, it could only fail to be differentiable at "a few" points in some sense, like with our examples above. However, Weierstrass produced an example of a function which is actually not differentiable at *any* real number, despite being continuous everywhere. It looks like this:[1]



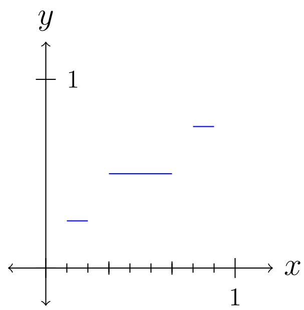
You can see how it's sort of "fractally pointy": at any point, no matter how smooth it may seem, you can zoom in enough to a point where it will look similar to the whole thing, and again become pointy at every point.

Another fractal-like example, this time in a different direction, is Cantor's function. It works like this. Consider just the interval from 0 to 1 for simplicity, though one can do it for all real numbers. Divide the interval into thirds, and draw in the middle third halfway between 0 and 1:

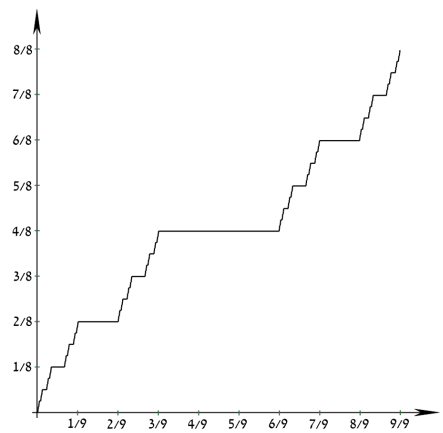
[1] Eeyore22, Public domain, via Wikimedia Commons.



For the first and last thirds, repeat the process:



Repeating indefinitely, this eventually comes to look like this:[2]



Miraculously, it is continuous, and its derivative exists at “almost every point” in a particular sense; when it exists, it is always 0! But infinitely often throughout this region the derivative fails to exist, despite the continuity of the function.

[2]CantorEscalier.svg: Theon derivative work: Amirki, CC BY-SA 3.0 `https://creativecommons.org/licenses/by-sa/3.0`, via Wikimedia Commons.