Lecture 18: the fundamental theorem of calculus

Calculus I, section 10
November 14, 2023

Last time, we introduced a new kind of calculus problem: evaluating definite integrals, which we think of as the "cumulative" value of a function so far, for example as in the area under a curve between two points. We saw how we can approximate integrals by Riemann sums, and rigorously define them as a limit; finally we hinted towards how one could compute them using indefinite integrals, also called antiderivatives. Today we'll see how to do this, using the fundamental theorem of calculus.

Let's first go back to the most general case we had completely worked out, that of linear functions, and make an observation: suppose we take the integral from 0 to a fixed point b,

$$\int_0^b (mx + k)\, dx.$$

From our calculation before, we know that in this case (with $a = 0$) this should be equal to $\frac{1}{2}mb^2 + kb$. This depends on the choice of the point b, and indeed we could think of it as a function of b; let's call this $F(b)$. Then observe: the derivative of F

$$F'(b) = \frac{1}{2}m \cdot 2b + k = mb + k$$

recovers the original function $f(x) = mx + k$, evaluated at b!

This is hinting at a more general principle, which will become the fundamental theorem of calculus. Before we get there, though, it motivates the notion of an *antiderivative*, also called an *indefinite integral* (in contrast with the definite integrals above). Given a continuous function $f(x)$, we say that another function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$. In the example above, we saw that $F(x) = \frac{1}{2}mx^2 + kx$ is an antiderivative of $f(x) = mx + k$. A special case is that if $m = 0$, so $f(x) = k$ is constant, then $F(x) = kx$ is an antiderivative; if $k = 0$ and $m = 1$, so $f(x) = x$, then $F(x) = \frac{1}{2}x^2$ is an antiderivative.

Why do we say "an" antiderivative rather than "the"? Because a function can have multiple antiderivatives, and indeed always will: if $F(x)$ is an antiderivative of $f(x)$, i.e. $\frac{d}{dx}F(x) = f(x)$, then so is $F(x)+C$ for any constant C, because $\frac{d}{dx}(F(x)+C) = F'(x) = f(x)$. We'll often write

$$\int f(x)\, dx$$

for an antiderivative of $f(x)$ (corresponding to the term "indefinite integral"), and we usually put

$$\int f(x)\, dx + C$$

to denote that this is really a *family* of functions, which differ by an additive constant.

What about other antiderivatives? Can we have two antiderivatives $F(x)$ and $G(x)$ which aren't the same up to an additive constant? No, by the mean value theorem! If

$F'(x) = G'(x) = f(x)$, then $0 = F'(x) - G'(x) = \frac{d}{dx}(F(x) - G(x))$, by linearity. But then one of our applications of the mean value theorem was that any function whose derivative is everywhere 0 must be constant, so it follows that $F(x) - G(x)$ must be a constant C, i.e. $F(x) = G(x) + C$. Thus the theory of antiderivatives is on a solid footing, with only this ambiguity of a constant. Using our knowledge of derivatives, we can compute some indefinite integrals; our goal is to see how we can use indefinite integrals to evaluate definite integrals.

Theorem (Fundamental theorem of calculus, first version)**.** *Let $f(x)$ be an integrable function on the interval $[a, b]$, and*

$$F(y) = \int_a^y f(x)\,dx$$

for any y between a and b. Then $\frac{d}{dy}F(y) = f(y)$, i.e. $F' = f$.

Already this generalizes the above observation, even when $f(x) = mx + k$, since this tells us that it's true even starting from any a, not just $a = 0$.

Nevertheless, although this is a cool observation it doesn't really help us much at first glance. It's nice to know the derivative of F, but really what we want is to be able to compute the value of F at particular points b, i.e. $F(b) = \int_a^b f(x)\,dx$.

What we can note, though, is that the theorem tells us that $F' = f$, or in other words that F is an antiderivative of f. This suggests that maybe if we first find an antiderivative of f, we can use it to compute integrals.

Let's try approaching the problem from this direction: say that we've found an antiderivative G of f, i.e. $G'(x) = f(x)$. We know from last time that this means F and G are the same, up to a constant:

$$F(y) = \int_a^y f(x)\,dx = G(y) + C$$

for some constant C. Since C doesn't depend on y, we can choose a value of y to make the calculation simple; a good choice is $y = a$, since then the integral is across the interval $[a, a]$ which is a single point, so the area must be 0. Thus we have

$$F(a) = 0 = G(0) + C,$$

so $C = -G(0)$. Therefore in general

$$F(y) = \int_a^y f(x)\,dx = G(y) - G(a).$$

This notation is a little redundant: we've wound up writing y instead of b, and G instead of F since we meant something else by F. Cleaning this up by replacing some of the variable and function names, we've found the following:

Theorem (Fundamental theorem of calculus, second version)**.** *Let $f(x)$ be an integrable function on the interval $[a, b]$, and $F(x)$ be an antiderivative of $f(x)$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

For example, recall the integral we were interested in last time:

$$\int_0^1 x^2\,dx.$$

We found some upper and lower bounds by hand using Riemann sums; taking the limit and using some trickery, we deduced that the answer should be $\frac{1}{3}$, but we didn't really have any kind of method that could be generalized. Now we do, so let's see if the answers agree: we can check that $\frac{x^3}{3}$ is an antiderivative of x^2, since $\frac{d}{dx}\frac{x^3}{3} = \frac{1}{3}\cdot 3x^2 = x^2$. (You'll generalize this calculation on the homework.) We find that

$$\int_a^b x^2\,dx = \left(\frac{x^3}{3}\right)\bigg|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

just as we predicted last time.

Another example which will appear on your homework is

$$\int_0^\pi \sin(x)\,dx.$$

One one part of your homework, you'll approximate this integral via Riemann sums. On another, you'll (hopefully) find that its antiderivative is $-\cos(x)$ (plus an additive constant), so

$$\int_0^\pi \sin(x)\,dx = (-\cos(x))|_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

Hopefully this should be reasonably close to the approximations you get (though not too close, since the problem only asks for four intervals).

Now that we've seen a little bit of how powerful this tool can be, let's think for a moment about why it might be true. Since we saw how the second version of the theorem follows from the first version, we'll focus on the first version.

By definition, $F'(y)$ is

$$\lim_{h\to 0}\frac{1}{h}\left(\int_a^{y+h} f(x)\,dx - \int_a^y f(x)\,dx\right).$$

There is a general property of definite integrals which will be useful to us here, and is worth writing out:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

because the area under the curve of $f(x)$ between a and b plus the area between b and c is just the area between a and c.

Applying this in our case, we get

$$F'(y) = \lim_{h\to 0}\frac{1}{h}\int_y^{y+h} f(x)\,dx.$$

There are a few ways we could proceed: one is to now replace the integral by a Riemann sum, and take the limit. Another is to say that because the interval $[y, y+h]$ has length h, which is going to 0 , we should actually be able to approximate the integral over this interval well by a single rectangle. This is equivalent to approximating $f(x)$ on this interval by a constant, say $f(y)$, so that we'd get a total area for the integral of $hf(y)$. When we divide by h, we conclude that $F'(y) = f(y)$, as desired.

If we're suspicious of this zeroth-order approximation, we could try using the first-order approximation instead: in the limit, we can safely replace $f(x)$ by $f(y)+(x-y)f'(y)$. Then the integral is

$$\int_y^{y+h} \left(f(y) + (x-y)f'(y)\right)\, dx$$

which is the definite integral of a linear function of x. We already know a formula for these integrals; I won't make you work through it, but it should come out to

$$\frac{1}{2}f'(y)h^2 + f(y)h.$$

Thus in total we get

$$F'(y) = \lim_{h\to 0} \frac{\frac{1}{2}f'(y)h^2 + f(y)h}{h} = \lim_{h\to 0} \left(\frac{1}{2}f'(y)h + f(y)\right) = f(y).$$

To see more examples, we'll have to think about some more antiderivatives. Let's start as usual with polynomials. On the homework, you'll work out how to antidifferentiate (and thus integrate) expressions of the form x^n:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

To combine them, we use another general property of integration: linearity. Just like for derivatives, for two integrable functions f and g we have

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

and for any constant c we get

$$\int cf(x)\, dx = c\int f(x)\, dx.$$

You can prove these by differentiating, and using the fact that differentiating is linear: for example, $\int cf(x)\, dx$ is a function whose derivative is $cf(x)$, and if $F(x) = \int f(x)\, dx$ is an antiderivative of $f(x)$ then $cF(x)$ is an antiderivative of $cf(x)$, so we get the identity above.

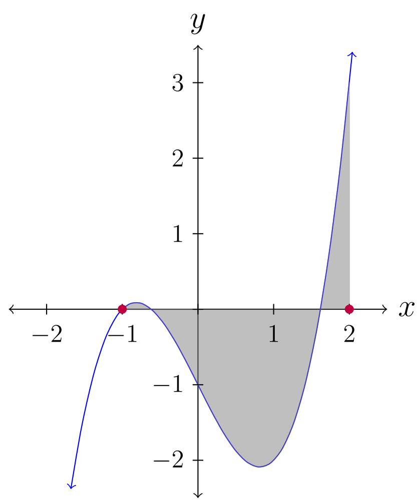
Using linearity for antiderivatives together with the fundamental theorem of calculus, we get the same linearity statement for definite integrals. (We could also have tried to prove it from the Riemann sum formula.) This should be pretty intuitive: if we add two functions vertically, we expect to add the corresponding areas, and similarly if we scale a function we should scale its area.

Applying linearity to monomials x^n, just like for differentiation this means we can now integrate any polynomial! For example,

$$\begin{aligned}\int_{-1}^{2} x^3 - 2x - 1\,dx &= \left(\frac{x^4}{4} - 2 \cdot \frac{x^2}{2} - x\right)\Big|_{-1}^{2} \\ &= \left(\frac{2^4}{4} - 2^2 - 2\right) - \left(\frac{(-1)^4}{4} - (-1)^2 - (-1)\right) \\ &= -2 - \frac{1}{4} \\ &= -\frac{9}{4}.\end{aligned}$$

Now let's take a moment to think about this. We did all our steps, and we got a negative result. But we said that the integral is supposed to give us an area; how can this be negative?

To see what's happened, let's try sketching the situation.



You can sort of see what's happening here: we have a little bit of area above the x-axis at first, then a lot below the x-axis, and then a little more above the x-axis. We consider the area above the x-axis to be positive, and below the x-axis to be negative. So what's happening here is that more of the total area is negative then positive, and so the total ends up negative.

Why do we count some of the area as negative? Because what the integral is about is not really area, although that's one convenient way to think about it: it's really about cumulative value. If the function is negative at a point, then that actually reduces the cumulative value, so we want to count that part as negative. If we care about actual area, rather than this sort of "signed" area, we should instead of $\int_a^b f(x)\,dx$ use $\int_a^b |f(x)|\,dx$.

Okay, so we understand how to integrate polynomials. Next comes rational functions, but integrating rational functions is actually quite hard in general (and is one of the important topics of calculus 2). One particular rational function that we'd like to be able to integrate is $f(x) = \frac{1}{x}$, because our previous method fails: this is x^{-1}, but its antiderivative can't be $x^{-1+1} - 1 + 1 = \frac{1}{0}$!

Instead, we recall a derivative: $\frac{d}{dx} \ln(x) = \frac{1}{x}$, so $\int \frac{1}{x}\, dx = \ln(x) + C$. In particular,

$$\int_1^y \frac{1}{x}\, dx = \ln(y) - \ln(1) = \ln(y)$$

and in fact this is one way to define $\ln(x)$. We can then define the exponential function e^x as the inverse function of $\ln(x)$, instead of the other way around, and prove its properties from there.

Next we have exponential functions themselves. Since e^x is its own derivative, we have

$$\int e^x\, dx = e^x + C.$$

How would you integrate a general exponential function?

You'll see some trigonometric antiderivatives on your homework; others are possible, and become important in calculus 2. One interesting one is that you might recall that $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2+1}$. This means that

$$\int_a^b \frac{1}{x^2+1} = \tan^{-1}(b) - \tan^{-1}(a).$$

For example, $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(0) = 0$, so

$$\int_0^1 \frac{1}{x^2+1}\, dx = \frac{\pi}{4}$$

or equivalently

$$\pi = \int_0^1 \frac{4}{x^2+1}\, dx.$$

Since the definite integral on the right is something we can numerically approximate (by Riemann sums or various improvements some of you suggested last class), we could actually use this formula to compute π !

Generally, the intuition is that integration is much harder than differentiation. We learned how to differentiate essentially any function we know how to write down in this course, but for integration the situation is much worse: there are many functions that are easy to write down but have no integral in terms of functions we know. For example, $\sin(\sin(x))$ has no antiderivative in terms of any functions we know, and in general we have no way of computing the corresponding integrals except numerically.[1] In particular, we don't have the rules of differentiation (such as the product, quotient, or chain rules), other than linearity: these were

[1] There may be special choices of bounds for which we can say exactly what the integral is: for example, if $a = -b$, then since $\sin(\sin(x))$ is an odd function we get that $\int_{-b}^{b} \sin(\sin(x))\, dx = 0$.

what allowed us to differentiate arbitrary functions, but aren't true for integrals. Instead, they become techniques of integration, which sometimes work and allow us to integrate more complicated functions than we would otherwise be able to, but still not every function. We'll see a first example of one of these next time: u-substitution, which is the integration analogue of the chain rule.