Lecture 19: u-substitution

Calculus I, section 10
November 16, 2023

We now know what integrals are and, roughly speaking, how we can approach them: the fundamental theorem of calculus lets us compute definite integrals using indefinite integrals, which we can study using our knowledge of differentiation. Today's goal is to introduce a tool for computing antiderivatives: u-substitution.

Previously, we only have a couple of tools for antiderivatives: there are some expressions (such as $2x$) which we can recognize as the derivative of some other function (in this case x^2); there are expressions which we can recognize as closely related to derivatives we know, such as x ($\frac{1}{2}$ of $2x$, and so the derivative of $\frac{1}{2}x^2$); and there are expressions which are combinations of these, and so which we can integrate by linearity (such as $3x^2+\sin(x)$, which is the derivative of $x^3 - \cos(x)$). What about an expression like $\sin(1-2x)$, or $\frac{x}{x^2-2}$?

The idea is to use a form of the chain rule. We know that

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x).$$

Therefore if we're given something of the form

$$\int f(g(x))g'(x)\,dx,$$

if we can find an antiderivative F of f then $F(g(x))$ has derivative $F'(g(x))g'(x) = f(g(x))g'(x)$, and so is the antiderivative we're looking for (up to an additive constant).

For example, consider

$$\int 2x \cdot \sin(x^2)\,dx.$$

If we write $f(x) = \sin(x)$ and $g(x) = x^2$, then this is

$$\int f(g(x))g'(x)\,dx.$$

Since $-\cos(x)$ is an antiderivative of $\sin(x)$, this means that

$$\int 2x \cdot \sin(x^2)\,dx = -\cos(x^2) + C,$$

which we can verify by computing $\frac{d}{dx} - \cos(x^2) = \sin(x^2) \cdot 2x$ by the chain rule.

To simplify the notation, we'll often introduce another variable, typically called u, which is why this method is called u-substitution. We set $u = g(x)$, and then employ another notational trick: recall we said that the dx in an integral is the same as in $\frac{d}{dx}$. We have several notations for the derivative: $\frac{d}{dx}g(x) = \frac{dg}{dx} = g'(x)$. Since these are all supposed to be the same thing, we should be able to "multiply through by dx" to get $dg = g'(x)\,dx$. Since we're setting $u = g(x)$, we write the left-hand side as $du = g'(x)\,dx$, and so

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Now this is just the antiderivative of f, which we assume we can compute.

In our example above, we set $u = x^2$, so $du = 2x\,dx$. Therefore

$$\int 2x \cdot \sin(x^2)\,dx = \int \sin(u)\,du = -\cos(u) + C.$$

This doesn't quite look like what we had before, but it's because now it's in terms of u; we want an answer in terms of our original variable, x. To get it, we just plug back in the definition of u: $u = x^2$, so the integral is $-\cos(x^2) + C$, just as before.

Let's try another example:

$$\int \sin(1 - 2x)\,dx,$$

which we mentioned above. A natural guess for u is $u = 1 - 2x$, so $du = \frac{d}{dx}(1 - 2x) \cdot dx = -2\,dx$.

We have a dx, but there's no -2 in sight! How can we proceed?

Well, we could insert a factor of -2, if we also divide by it:

$$\int \sin(1 - 2x)\,dx = \int \frac{1}{-2} \sin(1 - 2x) \cdot (-2)\,dx.$$

Now we can use the u-substitution: this is

$$\int \frac{1}{-2} \sin(u)\,du = -\frac{1}{2} \cdot (-\cos(u)) + C = \frac{1}{2} \cos(1 - 2x) + C.$$

Another way we could think about this process, rather than mysteriously dividing by -2, is that what we're really doing is solving for dx. In the previous example, we computed du and found that it was already present in the integral. Sometimes this will happen, and it's always nice when it does. When it doesn't, though, what we already have in the integral is a dx, and since we know the relationship between du and dx we can solve for dx and substitute the result. In this case, since $du = -2\,dx$ we have $dx = \frac{1}{-2} \cdot du$, and so

$$\int \sin(1 - 2x)\,dx = \int \sin(u) \cdot \frac{1}{-2}\,du,$$

and then we can proceed as above.

Note: you might be tempted to not mess with the dx and du business, and just substitute u and have done with it. This would give you

$$\int \sin(u)\,du = -\cos(u) + C = -\cos(1 - 2x) + C,$$

which is wrong! Keeping track of the difference between du and dx is important: it is analogous to the correction factor $g'(x)$ from the chain rule, and can't be left out.

The other example problem we mentioned above was

$$\int \frac{x}{x^2 - 2}\,dx.$$

We can now guess how we should approach it: set $u = x^2 - 2$. Then $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$, and so this is

$$\int \frac{x}{u} \cdot \frac{1}{2x}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln(x^2 - 2) + C.$$

Sometimes, we have to make multiple substitutions. Consider

$$\int \frac{e^{4x} + e^{3x} - e^x}{e^{3x} - 1}\,dx.$$

Since $e^{4x} = (e^x)^4$ and similarly for the other terms, a reasonable guess is $u = e^x$. This gives $du = e^x\,dx$, so $dx = \frac{1}{e^x}\,du = \frac{1}{u}\,du$ and so our integral is

$$\int \frac{u^4 + u^3 - u}{u^3 - 1} \cdot \frac{1}{u}\,du = \int \frac{u^3 + u^2 - 1}{u^3 - 1}\,du.$$

To proceed, we can notice that the numerator contains a copy of $u^3 - 1$:

$$\frac{u^3 + u^2 - 1}{u^3 - 1} = \frac{u^3 - 1}{u^3 - 1} + \frac{u^2}{u^3 - 1} = 1 + \frac{u^2}{u^3 - 1}.$$

By linearity, we can take this one term at a time: the antiderivative of 1 is just u, up to an additive constant, so we can just worry about

$$\int \frac{u^2}{u^3 - 1}\,du.$$

To find this integral, we use *another* substitution: $w = u^3 - 1$, so $dw = 3u^2\,du$, or $du = \frac{1}{3u^2}\,dw$. Thus our integral is

$$\frac{1}{3}\int \frac{1}{w}\,dw = \frac{1}{3}\ln(w) + C.$$

We can then substitute back the definition of w and combine with the integral of 1 to get $u + \frac{1}{3}\ln(u^3 - 1) + C$. But we want to be in terms of our original variable x, and so we again substitute the definition of u to get

$$\int \frac{e^{4x} + e^{3x} - e^x}{e^{3x} - 1}\,dx = e^x + \frac{1}{3}\ln(e^{3x} - 1) + C.$$

In fact, we could have made a similar observation earlier:

$$\frac{e^{4x} + e^{3x} - e^x}{e^{3x} - 1} = \frac{e^x \cdot (e^{3x} - 1) + e^{3x}}{e^{3x} - 1} = e^x + \frac{e^{3x}}{e^{3x} - 1}$$

and then substitute $u = e^{3x} - 1$. This gives $du = 3e^{3x}\,dx$ and so the integral is

$$\int e^x\,dx + \int \frac{1}{u}\,du = e^x + \frac{1}{3}\ln(u) + C = e^x + \frac{1}{3}\ln(e^{3x} - 1) + C,$$

as above. This is typical of this sort of situation: making a more complicated substitution earlier can often make the calculations faster and slicker, but it usually requires you to make some observation that is simpler to see when we expand things out via multiple substitutions.

Sometimes, picking the right substitution is not obvious either. Consider

$$\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

The intuition we've developed might suggest choosing $u = 1 - x^2$, so $du = -2x\,dx$. But there is no factor of x present, and so this would just turn into

$$\int \frac{1}{\sqrt{u}} \cdot \frac{1}{-2x}\,du.$$

Since $u = 1 - x^2$, we could write $x = \sqrt{1-u}$ (at least for $x \geq 0$) to get this as

$$-\frac{1}{2}\int \frac{1}{\sqrt{u}} \cdot \frac{1}{\sqrt{1-u}}\,du = -\frac{1}{2}\int \frac{1}{\sqrt{u \cdot (1-u)}}\,du,$$

but this is no easier.

Instead, we make a surprising substitution: set $u = \sin^{-1}(x)$. This is completely out of left field, but turns out to work well: by the inverse function rule, $du = \frac{d}{dx}\sin^{-1}(x)\,dx = \frac{1}{\sin'(\sin^{-1}(x)}\,dx = \frac{1}{\cos(\sin^{-1}(x))}\,dx = \frac{1}{\cos(u)}\,dx$, so $dx = \cos(u)\,du$ and $x = \sin(u)$. Therefore our integral is

$$\int \frac{1}{\sqrt{1-\sin(u)^2}} \cdot \cos(u)\,du = \int \frac{\cos(u)}{\cos(u)}\,du = \int 1\,du = u + C = \sin^{-1}(x) + C.$$

(We could also have seen this directly, by observing that $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$, but this requires less trigonometry, only the identity that $\sin(u)^2 + \cos(u)^2 = 1$.)

Let's now turn to applying this method to compute *definite* integrals. There are two ways to do this: one is to keep track of the endpoints as we go, the other is to forget them in the meantime and evaluate at the end.

Let's try both. Consider the integral

$$\int_{1/\pi}^{2/\pi} \frac{\sin(\frac{1}{x})}{x^2}\,dx.$$

If $u = \frac{1}{x}$, then $du = -\frac{1}{x^2}\,dx$, so we might be tempted to say this is

$$-\int_{1/\pi}^{2/\pi} \sin(u)\,du.$$

But be careful: we should think of the endpoints not just as numbers, but really as $x = 1/\pi$ and $x = 2/\pi$. Thus when we switch to u, we should change the endpoints as well: if $x = 1/\pi$, then $u = 1/x = \pi$, and similarly if $x = 2/\pi$ then $u = \pi/2$. Therefore in terms of u this is really

$$-\int_{\pi}^{\pi/2} \sin(u)\,du = \cos(\pi/2) - \cos(\pi) = 0 - (-1) = 1.$$

Notice, by the way, that there's something weird happening here: the lower bound is greater than the upper bound, i.e. we're integrating backwards! The effect is to flip the sign of the integral. One way to see this is that, by our rule from last time,

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0,$$

so

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$$

i.e. switching the endpoints reverses the sign.

Another approach is to back up and say first, we just look for an antiderivative of $\frac{\sin(\frac{1}{x})}{x^2}$, using u-substitution; then we apply the fundamental theorem of calculus at the end. Using the same substitution as above, we get that

$$\int \frac{\sin(\frac{1}{x})}{x^2}\,dx = -\int \sin(u)\,du = \cos(u) + C = \cos\left(\frac{1}{x}\right) + C,$$

substituting back in the definition of u. Now plugging in the original endpoints (since we're back in terms of x) gives

$$\cos\left(\frac{1}{2/\pi}\right) - \cos\left(\frac{1}{1/\pi}\right) = \cos(\pi/2) - \cos(\pi) = 1$$

as above.

Which method you prefer is a matter of personal taste; I find that I tend to get less confused using the second method, where we only plug in the endpoints at the end, but either will work.

Let's do one more:

$$\int_0^{10} (2x+1)^5\,dx.$$

We could expand out $(2x+1)^5$ as a big polynomial, but it would be very painful. Much easier is to substitute $u = 2x + 1$, so $du = 2\,dx$. The first method above would then turn into

$$\int_1^{21} u^5 \cdot \frac{1}{2}\,du = \left.\frac{u^6}{12}\right|_1^{21} = \frac{1}{12}(21^6 - 1) = \frac{21441530}{3}.$$

The second would tell us to first calculate

$$\int (2x+1)^5\,dx = \frac{1}{2}\int u^5\,du = \frac{u^6}{12} + C = \frac{(2x+1)^6}{12} + C,$$

and then evaluate at 10 and 1 to get

$$\frac{1}{12}(21^6 - 1)$$

as above.