

## Lecture 20: applications of integration

Calculus I, section 10

November 28, 2023

Welcome to our last regular lecture of the semester! By now, we have a pretty good understanding of definite and indefinite integrals, the relationship between them, and some techniques to calculate indefinite (and thus definite) integrals. Today, we'll switch focus a little and think about some applications of integrals, now that we can calculate them. (As with techniques of integration, this is only a small taste: calculus 2 or many other math or physics classes, among others, give many more examples of applications.)

On the worksheet, we looked at a common type of example in physics: if position is  $x(t)$ , velocity is  $v(t) = x'(t)$ , and acceleration is  $a(t) = v'(t) = x''(t)$ , then

$$v(t) = v(0) + \int_0^t a(s) ds$$

and

$$x(t) = x(0) + \int_0^t v(s) ds,$$

so if we know the initial position and velocity as well as the acceleration at all  $t$  we can compute the velocity and position at every time  $t$ . The necessary input is then given by Newton's second law  $F = ma$ , so by understanding the forces acting on an object we can understand its acceleration and thus its velocity and position. For example, if  $F$  and  $m$  are constant (as in the gravitational example), so that  $a = a(t) = \frac{F}{m}$  is constant, then

$$v(t) = v(0) + \int_0^t a ds = v(0) + at$$

and so

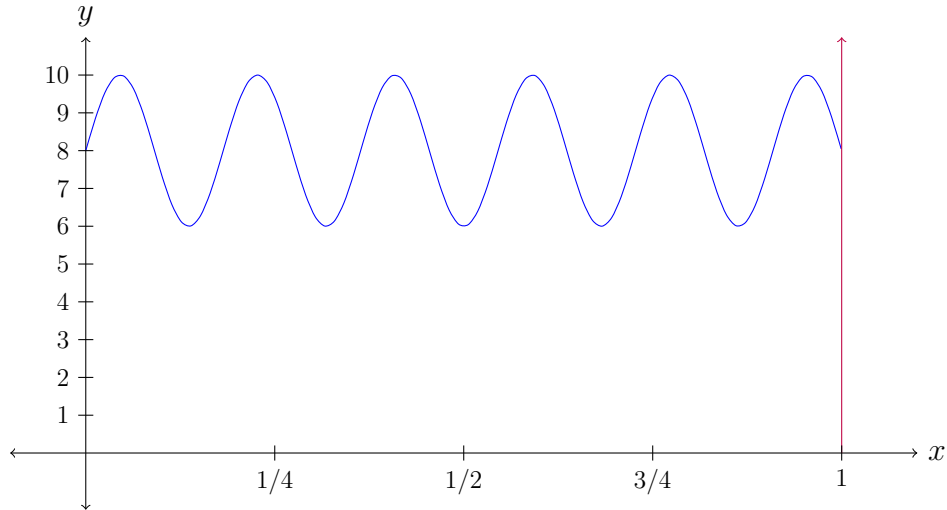
$$x(t) = x(0) + \int_0^t v(s) ds = x(0) + \int_0^t v(0) + as ds = x(0) + v(0)t + \frac{1}{2}at^2.$$

In the example where  $a = \frac{F}{m} = -g$  and  $x(0) = x_0$ ,  $v(0) = v_0$ , this recovers

$$x(t) = -\frac{1}{2}gt^2 + v_0t + x_0,$$

with  $g = 32$  feet per second squared recovering  $-16t^2 + v_0t + x_0$ .

Another application related to this perspective of cumulative value is finding the *average* value. Suppose that a runner is running for one hour; sometimes they run faster and sometimes slower, at a speed of  $v(t) = 8 + 2\sin(11\pi t)$ .



What is the average speed of the runner over the hour?

We don't really have good tools to think about averages of continuously changing quantities. However, as it turns out we don't have to: one simple way of measuring the average speed would be to figure out how far the runner ran would be if we knew how *far* they ran, and then we could simply divide the total distance by the total time (here one hour).

Fortunately, we just saw how to find the total distance they ran:

$$x(1) - x(0) = \int_0^1 v(t) dt = \int_0^1 8 + 2 \sin(11\pi t) dt.$$

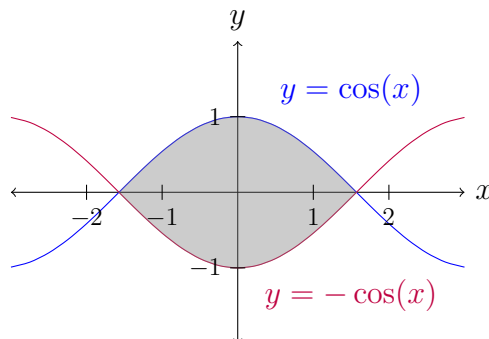
By linearity, this is

$$8 \cdot (1 - 0) + 2 \int_0^1 \sin(11\pi t) dt.$$

To conclude, we substitute  $u = 11\pi t$ , so  $du = 11\pi dt$ , so the integral becomes

$$8 + 2 \int_0^{11\pi} \sin(u) \cdot \frac{1}{11\pi} du = 8 + \frac{2}{11\pi} (-\cos(11\pi) + \cos(0)) = 8 + \frac{4}{11\pi} \approx 8.11575.$$

Another common application of integrals is what we originally introduced them for: finding areas. We've looked at finding areas under a given curve, i.e. between the curve and the line  $y = 0$ . Often, though, we're really interested in finding areas of shapes which aren't easily thought of in this way. For example, how would you find the area of this shape?



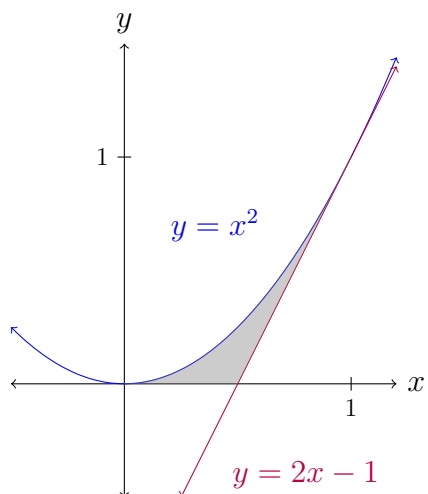
It is not the area beneath a curve; instead, it is the area between two curves!

To solve our problem, we think about integrating the *height* of the region. After all, this is what we're usually doing, it's just that the bottom of the region is usually at 0; this lets us add up the heights as usual to get the total area. Thus whenever we want to find the area between  $y = f(x)$  and  $y = g(x)$  on a certain region, if say  $f(x)$  is on top in that region then we integrate  $f(x) - g(x)$ . In this case,  $\cos(x)$  is greater than  $-\cos(x)$  in this region, so we're integrating  $\cos(x) - (-\cos(x)) = 2\cos(x)$ . (We could also have guessed this by observing that the  $x$ -axis divides the region into two identical regions, each of which looks like the integral of  $\cos(x)$ .)

Finding the bounds can also be a little tricky with these kinds of problems; they're not always given to us. Here, we're going between two points where the curves intersect, which is also at  $y = 0$ ; we have  $\cos(x) = -\cos(x) = 0$  at  $\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$ ,  $\frac{5\pi}{2}$ , and so on, and also at  $-\frac{\pi}{2}$ ,  $-\frac{3\pi}{2}$ , and so on. Here, the relevant points are  $\pm\frac{\pi}{2}$ , so our total area will be

$$\int_{-\pi/2}^{\pi/2} 2\cos(x) dx = 2\sin(\pi/2) - 2\sin(-\pi/2) = 4.$$

Another example is this: find the area of the shaded region.

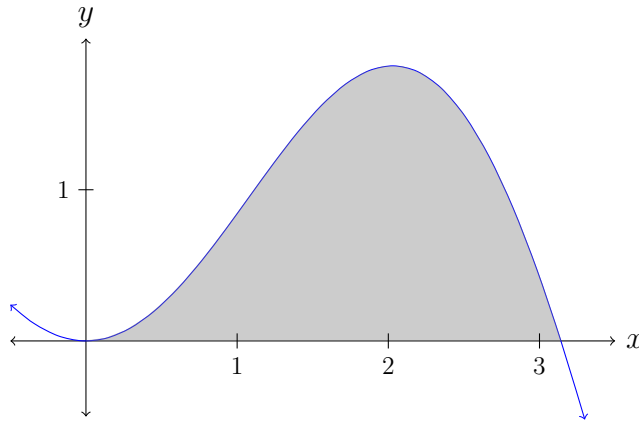


We could try to do the same thing. Now, though, the lower bound is piecewise: up to  $x = \frac{1}{2}$  it's just 0, but between  $\frac{1}{2}$  and the intersection of  $y = x^2$  and  $y = 2x - 1$  (which we can find is at  $x = 1$ ) it's  $y = 2x - 1$ . Thus we need to split the integral: the area is

$$\begin{aligned} \int_0^{1/2} x^2 dx + \int_{1/2}^1 x^2 - 2x + 1 dx &= \frac{1}{3}x^3 \Big|_0^{1/2} + \left( \frac{1}{3}x^3 - x^2 + x \right) \Big|_{1/2}^1 \\ &= \frac{1}{3} \left( \frac{1}{2} \right)^3 + \left( \frac{1}{3} \cdot 1^3 - 1^2 + 1 \right) - \left( \frac{1}{3} \left( \frac{1}{2} \right)^3 - \left( \frac{1}{2} \right)^2 + \frac{1}{2} \right) \\ &= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \\ &= \frac{1}{12}. \end{aligned}$$

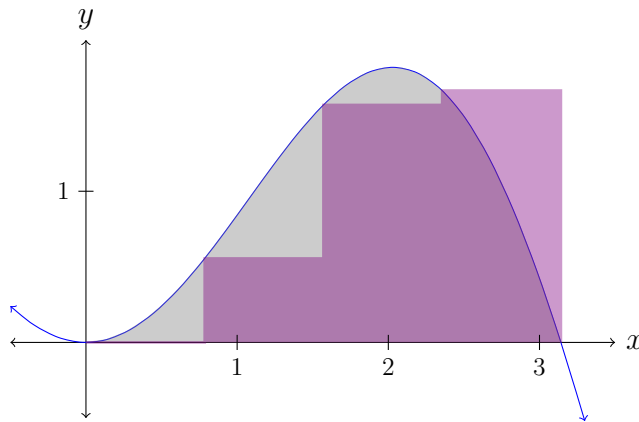
Finally, let's turn to the question of numerical approximation: how does one concretely compute integrals numerically, in cases where there isn't an exact answer or the exact answer is too complicated to be useful? How does your computer or calculator compute integrals?

We've already seen one way to compute integrals: via (either left or right) Riemann sums, which is how we defined them. Let's say we're trying to integrate something like  $x \sin(x)$  from 0 to  $\pi$ .

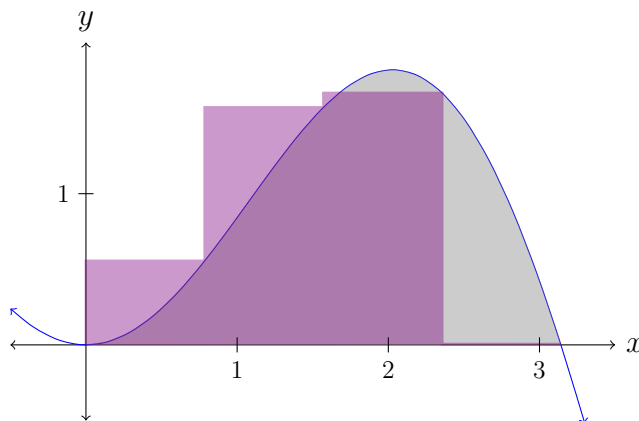


(This is possible to integrate exactly, but it's beyond the methods we've seen in this class; it would be done using integration by parts, and turns out to have value  $\pi$ .)

A four-step left Riemann sum would look like this:

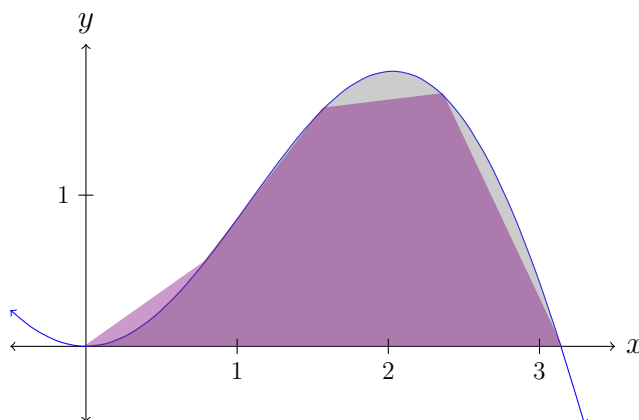


with total area estimate  $\approx 2.9784$ , while the right Riemann sum would look like this:



with the same area estimate (since both endpoints have value 0, so it's just the same area shifted).

For theoretical purposes, this is enough to define integrals, but for practical purposes we might want to do better. As suggested on a homework problem, we could do this by using nonzero slopes, i.e. replacing our rectangles with trapezoids. This is called the trapezoid rule.



This is still imperfect, but we can see it looks much closer, even with the same number of divisions. (In this case, as it turns out, it nevertheless actually gives exactly the same area estimate as the Riemann sums! This is somewhat special to this case, but isn't hugely unusual either: even though in each section the trapezoidal area is much closer to the real one, the errors in the Riemann sum method cancel out pretty well, so it's closer than it looks. Nevertheless in general the trapezoidal rule is much more reliable and converges much faster.)

An even more sophisticated approach is via Simpson's rule, which works by approximating the function by a *quadratic* interpolation between the endpoints, taking into account differential data. By using any given number of approximations, one can find convenient formulas; nevertheless in practice this turns out to be computationally more expensive than using the trapezoid rule or similar and just taking more intervals, which usually works better.

Finally, it's worth mentioning adaptive integration. Like for Newton's method, when we're trying to approximate integrals up to a given precision, say using Riemann sums or the trapezoid rule, we can just keep dividing into more and more intervals until the answer seems to be stabilizing: if the answer at  $N$  intervals and at  $N + 1$  intervals agree up to the allowed error, it's probably safe to stop. Adaptive integration takes this idea and goes further with it: in addition to checking the total error, we can also check the error on each interval. Thus if we're doing Riemann sums, regions in which the function is pretty close to constant don't really need very many intervals to get a pretty good approximation; but areas where the function is changing rapidly (i.e. has large derivative, in absolute value) need many more intervals. Similarly for the trapezoid rule, if the function is pretty close to linear the trapezoid rule will do pretty well even on relatively few intervals; but if it's far from linear, as near the maximum of  $x \sin(x)$  above, it needs more subdivisions to get a good approximation. In general, we can automate this process—and this is often what computers do—by setting an error threshold for each interval, and dividing each interval only until it

reaches that threshold; that way we put our resources where they will be most useful, and get a good approximation fastest.

One final application of integration, which we've touched upon before, is solving differential equations. As we've mentioned, this is a huge and complicated field; we'll only look at some simple examples. Nevertheless, integration (together with what we know about differential calculus) lets us solve some differential equations that we otherwise would not be able to solve. (These will not appear on the homework or exams.)

We can start by observing that simply finding antiderivatives is really already a kind of differential equation: e.g. finding the antiderivative of  $2x - 1$  is the same as solving the differential equation  $f'(x) = 2x - 1$ , which of course we can solve by integration:

$$f(x) = \int f'(x) dx = \int 2x - 1 dx = x^2 - x + C.$$

Note that the solution is determined up to an additive constant, which is familiar to us. If we wanted to specify one particular solution, we'd need some additional information; e.g. if we said  $f'(x) = 2x - 1$  and  $f(0) = 1$ , then since  $f(0) = 0^2 - 0 + C = C$  taking  $C = 1$  gives the unique solution  $f(x) = x^2 - x + 1$ .

For a more complicated differential equation, recall the example  $f'(x) = f(x)$ , a function which is its own derivative. You may recall from our differentiation unit that this is solved by  $f(x) = e^x$ ; it's also solved e.g. by  $f(x) = 0$ . Can we find all solutions in a systematic way?

One way to do so is to divide both sides by  $f(x)$ , so that we have  $\frac{f'(x)}{f(x)} = 1$ . This doesn't immediately look any easier, but we know a trick: logarithmic differentiation says that  $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$  (this is really a special case of the chain rule). So we have  $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)} = 1$ . Now we can integrate:

$$\ln f(x) = \int 1 dx = x + C,$$

so  $f(x) = e^{x+C}$ . Taking  $C = 0$  recovers  $e^x$ .

An interesting thing has happened here: the additive constant  $C$  has now gone from the outside of the expression into the exponent. By the rules of exponentiation,  $e^{x+C} = e^x \cdot e^C$ , so if we define a new constant  $K = e^C$  then we get  $f(x) = K \cdot e^x$  for a constant  $K$ ; so now instead of an additive constant, we have a multiplicative one. This sort of transformation is typical in differential equations: the solution usually depends on a parameter (or even multiple, for more complicated equations), but often in a more complex way than occurs for antidifferentiation. We can specify particular solutions again by initial data: for example, requiring  $f(0) = 1$  gives  $Ke^0 = K = 1$ , so  $f(x) = e^x$  is the unique solution to  $f'(x) = f(x)$  and  $f(0) = 1$ .

Finally, there's one other strange thing:  $f(x) = Ke^x$  is a full family of solutions, and all solutions will be of this form. However, this includes examples which we couldn't actually have obtained in our original form  $f(x) = e^{x+C}$ ! For example, taking  $K = 0$  gives  $f(x) = 0$ , which we mentioned before and which is clearly a solution to the differential equation. But  $K = e^C$  can never be zero! Similarly, negative values of  $K$  work just as well as positive ones; but  $e^C$  can never be negative.

This has to do with the fact that although it's true that  $\frac{d}{dx} \ln(x) = \frac{1}{x}$  for  $x > 0$ , it can't be true for  $x < 0$  simply because  $\ln(x)$  is then not defined; so really we should take  $\ln(|x|)$ . This then means taking  $K = \pm e^C$ , so that negative values are allowed. (This is also closely related to complex numbers and Euler's formula.) The case  $K = 0$  then arises as a limit of the other solutions.

As an exercise, if there's time: try working out the solutions to  $f'(x) = \frac{1}{f(x)}$ .