L-parameters and Hodge structures

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Last time, we introduced the stack $\mathrm{Div}^1 \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus 0)/W_{\mathbb{R}}^{\mathrm{la}}$ and its double cover $\mathrm{Div}^1_{\mathbb{C}} \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus 0)/\mathbb{C}^{\times,\mathrm{la}}$, and observed that on the open locus $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{C}^{\times}_{\mathrm{Betti}} \subset \mathrm{Div}^1_{\mathbb{C}}$ vector bundles are determined by their monodromy and correspond to representations of $\mathbb{C}^{\times,\mathrm{la}}$. Our first goal today is to prove a result stated last time: every vector bundle on $\mathrm{Div}^1 \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus 0)/W_{\mathbb{R}}^{\mathrm{la}}$ extends uniquely to $\mathbb{A}^{2,\mathrm{an}}/W_{\mathbb{R}}^{\mathrm{la}}$, so that the zero section and the projection to a point give an embedding of isomorphism classes of locally analytic $W_{\mathbb{R}}$-representations into vector bundles on Div^1, which is a bijection on semisimple objects. This justifies our perspective that vector bundles, or more generally $\widehat{G}$-torsors, on Div^1 should be our geometric notion of L-parameters in this setting; we will see that this actually gives rise to some objects not appearing in the classical picture, which makes the archimedean L-parameters better-behaved.

Part of this will involve relating vector bundles on certain related stacks to vector bundles with T-connection, which can be thought of as a variant of the de Rham stack construction. This leads to an interpretation of vector bundles on Div^1 in terms of Hodge structures. It is then of interest to generalize this picture, to understand variations of Hodge structures in this language; this is related to (the archimedean analogue of) diamondization, which time permitting we will introduce, though a detailed study will have to wait until next time.

1. Vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$

Last time, we saw that on the large open subset $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{C}^{\times}_{\mathrm{Betti}} \subset \mathrm{Div}^1_{\mathbb{C}}$ vector bundles are given by representations of $\mathbb{C}^{\times,\mathrm{la}}$, classified by their monodromy $\alpha \in \mathrm{GL}_n$ which can be described directly in terms of the representation in question. The difference between $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times}$ and $\mathrm{Div}^1_{\mathbb{C}} \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/\mathbb{C}^{\times,\mathrm{la}}$, which we think of as roughly a complex projective line, is given by two copies of $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}}$, which we can think of as the points at 0 and ∞. We look at adding back these missing points one at a time; but in fact they are interchanged under complex conjugation (which recall on this version of the projective line is really $z \mapsto -1/\overline{z}$), so it suffices to look at a neighborhood of ∞, i.e. adding back in the axis on the first coordinate, so $(\mathbb{A}^{1,\mathrm{an}} \times \mathbb{G}_{\mathrm{m}}^{\mathrm{an}})/\mathbb{C}^{\times,\mathrm{la}}$, with the scalar action twisted by complex conjugation on the second factor.

The action of $\mathbb{C}^{\times,\mathrm{la}}$ on $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$ is transitive, with stabilizer $\mathbb{G}_{\mathrm{m}}^{\dagger}$, so that the total quotient is

$$(\mathbb{A}^{1,\mathrm{an}} \times \mathbb{G}_{\mathrm{m}}^{\mathrm{an}})/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_{\mathrm{m}}^{\dagger}.$$

Puncturing $\mathbb{A}^{1,\mathrm{an}}$ at the origin yields $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}/\mathbb{G}_{\mathrm{m}}^{\dagger} \simeq \mathbb{C}^{\times}_{\mathrm{Betti}}$, the open subset we saw before. Thus we want to understand vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_{\mathrm{m}}^{\dagger}$. If we were quotienting by $\mathbb{G}_{\mathrm{a}}^{\dagger}$ instead, this would be the analytic de Rham stack $\mathbb{A}^{1,\mathrm{an}}_{\mathrm{dR}}$, vector bundles on which are equivalent to vector bundles on $\mathbb{A}^{1,\mathrm{an}}$ together with a connection; working through a similar argument here, adjusting to complete at the multiplicative identity in $\mathbb{A}^{1,\mathrm{an}}$ rather than the additive identity, gives the following result.

Proposition 1. *Vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger$ are equivalent to vector bundles E on $\mathbb{A}^{1,\mathrm{an}}$ together with a T-connection, i.e. a map $\Delta_E : E \to E \otimes \Omega^1$ such that for $f \in \mathcal{O}$ and $v \in E$ we have*

$$\Delta_E(fv) = f\Delta_E(v) + T\Delta(f)v$$

where T is the standard coordinate on $\mathbb{A}^{1,\mathrm{an}}$.

The analyticity is encoded via working over $\mathbb{A}^{1,\mathrm{an}}$ and so these are also equivalent to vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\widehat{\mathbb{G}_\mathrm{m}}$.

This is already enough to describe vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$: this was for the neighborhood around ∞, but the same description applies to the neighborhood around 0, so vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ are equivalent to pairs of vector bundles with T-connection (E_1, Δ_{E_1}), (E_2, Δ_{E_2}) on $\mathbb{A}^{1,\mathrm{an}}$, whose restriction to the overlap $\mathbb{G}_\mathrm{m}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{C}^\times_{\mathrm{Betti}}$ agree, i.e. with the same monodromy around 0.

Given a filtration on the restriction of a vector bundle on $\mathrm{Div}^1_{\mathbb{C}}$ to $\mathbb{G}_\mathrm{m}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}}$, extending the filtration to the whole space is then just taking the corresponding filtration on the E_i, so every filtration on the restriction extends uniquely to the whole space. In particular the decomposition of the category of vector bundles on $\mathbb{G}_\mathrm{m}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}}$ by the generalized eigenvalue of the monodromy extends to the category of vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$, with semisimple vector bundles again corresponding to semisimple monodromy and vice versa.

In particular, when α is semisimple by taking direct summands and twisting we can assume that it is trivial, so it is interesting to study vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ with trivial monodromy. We approach via the following result:

Proposition 2. *The category of vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger$ with trivial monodromy, i.e. vector bundles with T-connection on $\mathbb{A}^{1,\mathrm{an}}$ with trivial monodromy, are equivalent to vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}}$, and thence equivalent to filtered vector spaces (over $\mathbb{C}_{\mathrm{gas}}$).*

Proof. The first part is the statement that having trivial monodromy is equivalent to descending along the map

$$\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_\mathrm{m}^\dagger \to \mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_\mathrm{m}^{\mathrm{an}}.$$

This map is a torsor under $\mathbb{G}_\mathrm{m}^{\mathrm{an}}/\mathbb{G}_\mathrm{m}^\dagger \simeq \mathbb{G}_{\mathrm{m,dR}}^{\mathrm{an}} \simeq \mathbb{C}^\times_{\mathrm{Betti}}$, and descent along a $\mathbb{C}^\times_{\mathrm{Betti}}$-torsor is in fact equivalent to having trivial monodromy. Since vector bundles on $\mathbb{A}^1/\mathbb{G}_\mathrm{m}$ (the algebraic version) are known to be equivalent to filtered vector spaces (via the Rees module construction), to finish the theorem it suffices to show that pullback along the map

$$\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_\mathrm{m}^{\mathrm{an}} \to \mathbb{A}^1/\mathbb{G}_\mathrm{m}$$

induces an exact equivalence on vector bundles, which one can check by computing that the irreducible objects on each side are line bundles for which pullback gives a bijection and that the Hom and Ext groups agree. □

Using this fact for the neighborhoods of 0 and ∞, we deduce that vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ with trivial monodromy are equivalent to complex vector spaces equipped with two $\mathbb{C}$-filtrations. These are precisely complex Hodge structures, and one can show (via a similar stacky argument) that they all decompose into direct sums of one-dimensional complex Hodge structures, which are then classified by the degrees $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ of the filtration.

(Usually, we expect to identify one of these filtrations with the complex conjugate of the other; this is the real version, induced here by Galois descent.) We can equivalently think of these as complex numbers such that

$$\exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2) = 1,$$

which is precisely the statement that the monodromy α is trivial as expected.

Thus we have proven that vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ with semisimple monodromy are direct sums of line bundles which arise via pullback along

$$\mathrm{Div}^1_{\mathbb{C}} \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/\mathbb{C}^{\times,\mathrm{la}} \to */\mathbb{C}^{\times,\mathrm{la}}$$

from characters of $\mathbb{C}^\times$. In particular since this projection factors through $\mathbb{A}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}}$, every line bundle and thus every vector bundle with semisimple monodromy extends uniquely to $\mathbb{A}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}}$.

Being a little more careful, one can show that the inclusion

$$\mathrm{Div}^1_{\mathbb{C}} \hookrightarrow (\mathbb{A}^{2,\mathrm{an}} \times_{\mathbb{A}^2} (\mathbb{A}^2 \setminus \{0\}))/\mathbb{C}^{\times,\mathrm{la}}$$

induces via pullback an exact equivalence of categories of vector bundles. On the right, extension of vector bundles in the algebraic setting (over a codimension 2 subset) induces an equivalence with vector bundles on $\mathbb{A}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}}$, so this extends the above to the non-semisimple case. Finally everything descends along $\mathbb{C}/\mathbb{R}$, giving the result for vector bundles on Div^1 we claimed last time: every vector bundle on Div^1 extends uniquely to $\mathbb{A}^{2,\mathrm{an}}/W^{\mathrm{la}}_{\mathbb{R}}$, and the zero section and projection to a point of $\mathbb{A}^{2,\mathrm{an}}$ induce a factorization of the identity

$$\mathrm{Rep}(W^{\mathrm{la}}_{\mathbb{R}}) \xrightarrow{\pi^*} \mathbf{Vect}(\mathbb{A}^{2,\mathrm{an}}/W^{\mathrm{la}}_{\mathbb{R}}) \simeq \mathbf{Vect}(\mathrm{Div}^1) \xrightarrow{s^*} \mathrm{Rep}(W^{\mathrm{la}}_{\mathbb{R}})$$

so that π^* is injective on isomorphism classes, and if a vector bundle V on Div^1 is semisimple then it has semisimple monodromy and therefore is in the image of π^*, so isomorphism classes of semisimple $W^{\mathrm{la}}_{\mathbb{R}}$-representations are in bijection with isomorphism classes of semisimple vector bundles on Div^1. Note that there do exist however non-semisimple vector bundles which do not arise via pullback from $*/W^{\mathrm{la}}_{\mathbb{R}}$.

2. Relation to Adams–Barbasch–Vogan parameters

Assume for simplicity that $G/\mathbb{R}$ is split, so that L-parameters are just maps $W_{\mathbb{R}} \to \widehat{G}(\mathbb{C})$. Fix $\alpha \in \widehat{G}(\mathbb{C})$, and consider the moduli space of $\widehat{G}$-local systems on Div^1 together with a trivialization at the image of the point

$$1 \in \mathbb{C}^\times_{\mathrm{Betti}} \simeq \mathbb{G}_m^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \subset \mathrm{Div}^1_{\mathbb{C}}$$

after taking the cover $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1$ and resulting monodromy α. On $\mathrm{Div}^1_{\mathbb{C}}$, this is a pair of vector bundles whose restrictions to $\mathbb{C}^\times_{\mathrm{Betti}}$ agree and correspond to a local system with monodromy α; the Galois descent along $\mathbb{C}/\mathbb{R}$ interchanges the vector bundles with T-connection, so we are left with just a vector bundle with T-connection Λ on $\mathbb{A}^{1,\mathrm{an}}$, or equivalently vector bundle on $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger$, whose restriction to $\mathbb{G}_m^{\mathrm{an}}/\mathbb{G}_m^\dagger \simeq \mathbb{C}^\times_{\mathrm{Betti}}$ is a local system with monodromy

α; together with a Galois descent datum, which now just amounts to a square root y of the monodromy α. Allowing α to vary, we arrive at a space of pairs (Λ, y) with the monodromy of the restriction of Λ (as defined above) given by y^2.

We now want to identify this space of pairs (Λ, y) with the geometric parameter space of Adams–Barbasch–Vogan. We first need to introduce some notation: for any complex reductive group H with Lie group $\mathfrak{h}$, for a semisimple element $\lambda \in \mathfrak{h}$ and $n \in \mathbb{Z}$ we let

$$\mathfrak{h}(\lambda)_n = \{\mu \in \mathfrak{h} : [\lambda, \mu] = n\mu\},$$

the "λ-eigenspace" with eigenvalue n, and let

$$\mathfrak{n}(\lambda) = \sum_n \mathfrak{h}(\lambda)_n.$$

The canonical flat $\mathcal{F}(\lambda)$ through λ is the affine subspace

$$\mathcal{F}(\lambda) = \lambda + \mathfrak{n}(\lambda),$$

and write $\mathcal{F}(\mathfrak{h})$ for the set of all canonical flats. These give a partition of the semisimple elements of $\mathfrak{h}$, and the exponential map $e : \lambda \mapsto \exp(2\pi i\lambda)$ is constant on each canonical flat, so we can write $e(\Lambda) = e(\lambda)$ where $\Lambda = \mathcal{F}(\lambda)$ since the result only depends on Λ and not the choice of λ.

For $G = \mathrm{GL}_n$, we have seen before that the choice of a semisimple $n \times n$ matrix λ with $\exp(2\pi i\lambda) = \alpha$ can be viewed as a representation of $\mathbb{C}^{\times,\mathrm{la}}$ which we can pull back to a vector bundle on $\mathrm{Div}^1_{\mathbb{C}}$ with monodromy α plus a Galois descent datum, and by unique extension all such vector bundles arise in this way. We can do a version of this more generally for G: working out the story analogously, for any semisimple element $\lambda \in \widehat{G}(\mathbb{C})$ we get a $\widehat{G}$-local system on $\mathrm{Div}^1_{\mathbb{C}}$ with monodromy $e(\lambda)$ plus a Galois descent datum, and all such objects arise in this way. As above, this is equivalent to a $\widehat{G}$-local system on $\mathbb{A}^{1,\mathrm{an}}$ together with a T-connection and a square root y of $e(\lambda)$. Finally, we claim that the resulting object, like $e(\lambda)$, is independent of the choice $\lambda \in \Lambda = \mathcal{F}(\lambda)$; thus we get a natural bijection between $\widehat{G}$-local systems on Div^1 with semisimple monodromy and pairs (Λ, y) where $\Lambda \in \mathcal{F}(\widehat{\mathfrak{g}})$ and $y \in \widehat{G}(\mathbb{C})$ such that $e(\Lambda) = y^2$, i.e. Adams–Barbasch–Vogan L-parameters. If we require Λ to be contained within a single $\widehat{G}$-orbit of semisimple elements in $\widehat{\mathfrak{g}}$, then this has the structure of a smooth complex variety; in our generality we should view this as a component of the stack of L-parameters. (In this nonsplit case, we have to be a little more careful about the Langlands dual vs. the L-group, but the essential idea is the same.)

We digress briefly to discuss the motivation for these parameters. Last time, we mentioned that the standard notion of L-parameters in the archimedean setting do not behave very well in families: representations which can be deformed into each other may have L-parameters lying in different components of the parameter space. A more straightforward issue is the existence of L-packets: for GL_n, we have the pleasant phenomenon that the local Langlands correspondence gives a bijection between suitable representations and L-parameters, but for general groups G it is only finite-to-one, and we might speculate about a refinement that might produce a genuine bijection. An early realization of Vogan is that one should take all inner forms together (as we'll see is realized by the decomposition of

Bun_G). This already suggests parametrizing by classes in $H^1(\mathrm{Gal}_F, G)$ (for any local field F), or more generally $H^1(\mathcal{E}, G)$ for Galois gerbes $\mathcal{E}$, i.e. extensions of Gal_F by some group u. This gives rise to Kaletha's refined local Langlands correspondence, taking either u trivial, the finite adeles, or their integral subring; the last of these produces spaces of L-parameters which can be identified with Adams–Barbasch–Vogan L-parameters in the real case, but for our purposes the second is more natural, with the resulting gerbe corresponding to the Tannakian category of isocrystals and $H^1(\mathcal{E}, G) \simeq B(F, G)$. It seems very interesting to give a full geometrization of this picture in the archimedean setting; in the p-adic setting this is implicit in e.g. Kottwitz's work with Hansen and Weinstein on the Kottwitz conjecture.

3. Local duality on Div^1

Interpreting Weil group representations via Div^1 has another benefit: it lets us give an analogue of local Tate–Nakayama duality in the archimedean setting. For nonarchimedean local fields F, we can think of this local duality as the statement that the Weil group W_F has cohomological dimension 2, acting on ℓ-adic vector spaces, and satisfies an analogue of 2-dimensional Poincaré duality. We can interpret representations of W_F via the Fargues–Fontaine curve (or more precisely via the (nonarchimedean) mirror curve Div^1), which has cohomological dimension 2 and for which this is essentially literal Poincaré duality.

At infinity, though, this fails classically: $W_\mathbb{R}$ acting on $\mathbb{R}$-vector spaces has cohomological dimension 1, and does not have any apparent duality. Geometrically, though, we still have Div^1, and we get the following incarnation of duality:

Theorem 3. *The analytic stack*

$$f : \mathrm{Div}^1 \to \mathrm{AnSpec}\, \mathbb{C}_{\mathrm{gas}}$$

is proper and cohomologically smooth, with $f^!1 \simeq |\cdot|[2]$ where $|\cdot|$ is the line bundle on Div^1 corresponding to the norm character $|\cdot| : W_\mathbb{R} \to \mathbb{R}_{>0}$, given on $\mathbb{C}^\times \subset W_\mathbb{R}$ by $z \mapsto z\bar{z}$.

It formally follows from the yoga of six functor formalisms that there is a perfect pairing of finite-dimensional $\mathbb{C}$-vector spaces

$$H^i(\mathrm{Div}^1, E) \times H^{2-i}(\mathrm{Div}^1, E^\vee \otimes |\cdot|) \to H^2(\mathrm{Div}^1, |\cdot|) \simeq \mathbb{C}$$

for each vector bundle E on Div^1 and $0 \leq i \leq 2$, with higher cohomology vanishing.

Proof. There are three things to prove: properness, cohomological smoothness, and the identification of the dualizing complex. For the first two we can work with $\mathrm{Div}^1_\mathbb{C} \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/\mathbb{C}^{\times,\mathrm{la}}$, and then pass to Div^1 by Galois descent.

Thinking of $\mathrm{Div}^1_\mathbb{C}$ again as a version of the projective line, we can cover it by two copies of the overconvergent unit disk $\mathbb{D}^\dagger$ up to nil-scaling, i.e. $\mathbb{D}^\dagger/\mathbb{G}_\mathrm{m}^\dagger$, glued along $z \mapsto 1/\bar{z}$. The disks overlap along $|z| = 1$, i.e. the (analytic) circle group $\mathrm{U}(1)^{\mathrm{an}}$, so the two covering spaces $\mathbb{D}^\dagger/\mathbb{G}_\mathrm{m}^\dagger$ intersect at $\mathrm{U}(1)^{\mathrm{an}}/\mathbb{G}_\mathrm{m}^\dagger \simeq \mathrm{U}(1)^{\mathrm{an}}/\mathrm{U}(1)^\dagger \simeq \mathrm{U}(1)_{\mathrm{Betti}}$. Since $\mathbb{D}^\dagger \to *$ is proper, it remains to show that $p : */\mathbb{G}_\mathrm{m}^\dagger \to *$ is proper, so that the composite $\mathbb{D}^\dagger/\mathbb{G}_\mathrm{m}^\dagger \to */\mathbb{G}_\mathrm{m}^\dagger \to *$ will be; and for this it in turn suffices to show that $p_!1 \to p_*1$ is an isomorphism. For

$q : * \to */\mathbb{G}_m^\dagger$ the (proper) quotient map, we have $p_! q_* 1 = p_! q_! 1 = \mathrm{id}_! 1 = \mathrm{id}_* 1 = p_* q_* 1$ and 1 is a quotient of $q_* 1$, so the claim follows.

For properness, $\mathrm{Div}^1_{\mathbb{C}}$ is locally $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger$, whose cohomological smoothness follows from that of $\mathbb{A}^{1,\mathrm{an}}$ and $*/\mathbb{G}_m^\dagger$ (shown in a previous lecture). In fact this even shows that the cohomological dimension is 2, since it is for $\mathbb{A}^{1,\mathrm{an}}$ (together with the fact that $*/\mathbb{G}_m^\dagger$ has cohomological dimension 0, also shown previously).

Finally, we know that the dualizing complex must live in degree 2, where it is given by a character of $W_{\mathbb{R}}$ or equivalently of its abelianization $\mathbb{R}^\times$. Working out which character can be done via some explicit computation of Ext groups; we observe that the norm character is compatible (with respect to the projection $\mathrm{Div}^1 \to */W_{\mathbb{R}}^{\mathrm{la}}$) with the identification of the dualizing complex of $*/G^{\mathrm{la}}$ with the modulus character of G for any real Lie group G. □

4. Hodge structures

We saw above that vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ with trivial monodromy after restriction were equivalent to complex Hodge structures. More generally though vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ are more exotic objects, allowing for nontrivial monodromy. The centrality of the notion of (variations of) Hodge structures suggests that we should see if we can find a modification of $\mathrm{Div}^1_{\mathbb{C}}$ where vector bundles are actually equivalent to Hodge structures.

Recall that $X_{\mathbb{R}}$ admits an action of $\mathrm{O}(2)$ stabilizing the point at infinity, where it acts on the residue field $\mathbb{C}$ by the component map $\mathrm{O}(2) \to \mathbb{Z}/2 \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In fact, it acts in the relative setting as well: more precisely, the definition of $X_{\mathbb{R},A}$ descends to $*/\mathrm{O}(2)_{\mathrm{Betti}}$. This lets the whole theory descend to $*/\mathrm{O}(2)_{\mathrm{Betti}}$, and in particular $\mathrm{Div}^1_{\mathbb{C}}$ and Div^1 descend to $*/\mathrm{O}(2)_{\mathrm{Betti}}$.

Note that $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/\mathrm{O}(2)_{\mathrm{Betti}}$ is a gerbe over $\mathrm{AnSpec}\,\mathbb{R}_{\mathrm{gas}}$ banded by $\mathrm{U}(1)_{\mathrm{Betti}}$ (in fact the trivial gerbe). We are used to working with $\mathbb{C}$-coefficients rather than $\mathbb{R}$, so we will often only use the $\mathrm{U}(1)$-part of the $\mathrm{O}(2)$-action.

Working now with $\mathrm{Div}^1_{\mathbb{C}}/\mathrm{U}(1)_{\mathrm{Betti}} \to */\mathrm{U}(1)_{\mathrm{Betti}}$, we get an open subset $\mathbb{C}^\times_{\mathrm{Betti}}/\mathrm{U}(1)_{\mathrm{Betti}} \simeq \mathbb{R}_{>0,\mathrm{Betti}}$, which is (the realization of) a contractible manifold. Thus vector bundles on this open subset are equivalent to finite-dimensional $\mathbb{C}$-vector spaces. As previously, it remains to study the neighborhoods of 0 and ∞, and we can take these one at a time (they will be exchanged under the descent datum to $\mathbb{R}$): in a neighborhood of ∞, $\mathrm{Div}^1_{\mathbb{C}}$ is locally $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger \simeq \mathbb{A}^{1,\mathrm{an}}/\mathrm{U}(1)^\dagger$, so quotienting by $\mathrm{U}(1)_{\mathrm{Betti}} \simeq \mathrm{U}(1)^{\mathrm{la}}/\mathrm{U}(1)^\dagger$ gives $\mathbb{A}^{1,\mathrm{an}}/\mathrm{U}(1)^{\mathrm{la}}$. This is an analytic version of $\mathbb{A}^1/\mathbb{G}_m$, and an analogous argument to the algebraic case shows that vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\mathrm{U}(1)^{\mathrm{la}}$ are equivalent to filtered $\mathbb{C}$-vector spaces. Gluing with the analogous argument near 0, we arrive at the following proposition:

Proposition 4. *Vector bundles on* $\mathrm{Div}^1_{\mathbb{C}}/\mathrm{U}(1)_{\mathrm{Betti}}$ *are equivalent to finite-dimensional vector spaces* V *equipped with two (separated, exhaustive, decreasing) filtrations* $\mathrm{Fil}^\bullet V$, $\overline{\mathrm{Fil}}^\bullet V$, *i.e. complex Hodge structures.*

Similarly, after Galois descent we identify vector bundles on $\mathrm{Div}^1/\mathrm{U}(1)_{\mathrm{Betti}}$ with real Hodge structures.

In fact, the gluing description of this space via copies of $\mathbb{A}^{1,\mathrm{an}}/\mathrm{U}(1)^{\mathrm{la}}$ also yields the following isomorphism:

Proposition 5. *There is an isomorphism of analytic stacks*

$$\mathrm{Div}^1_{\mathbb{C}} \,/\, \mathrm{U}(1)_{\mathrm{Betti}} \simeq \mathbb{P}^1_{\mathbb{C}} / \,\mathrm{U}(1)^{\mathrm{la}}$$

over $\mathbb{C}_{\mathrm{gas}}$.

If we took the full O(2)-action rather than just U(1), i.e. incorporating the Galois descent, we would obtain

$$\mathrm{Div}^1 \,/\, \mathrm{U}(1)_{\mathrm{Betti}} \simeq X_{\mathbb{R}} / \,\mathrm{U}(1)^{\mathrm{la}}.$$ [1]

These justify our heuristics that $\mathrm{Div}^1_{\mathbb{C}}$ is "close to" the projective line while Div^1 is "close to" the (absolute) twistor $\mathbb{P}^1$.

In particular this gives a familiar $\mathrm{U}(1)_{\mathrm{Betti}}$-torsor $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}} \,/\, \mathrm{U}(1)_{\mathrm{Betti}}$ and further a $\mathrm{U}(1)^{\mathrm{la}}$-torsor $\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}} \,/\, \mathrm{U}(1)_{\mathrm{Betti}}$, which we will see is related to torsors. There is a map $\mathrm{U}(1)^{\mathrm{la}} \to \mathrm{U}(1)_{\mathrm{Betti}}$, along which we could push out the second torsor to get a $\mathrm{U}(1)_{\mathrm{Betti}}$-torsor over $\mathrm{Div}^1_{\mathbb{C}} \,/\, \mathrm{U}(1)_{\mathrm{Betti}}$; we might guess that in fact this is $\mathrm{Div}^1_{\mathbb{C}}$, but while this is true locally it turns out to fail globally: there is no map $\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}$ inducing the expected map on $\mathbb{C}$-points (it would have to be holomorphic at ∞ and anti-holomorphic at 0). This can be thought of as the failure of the archimedean Fargues–Fontaine curve (over $\mathbb{C}$) to literally be $\mathbb{P}^1_{\mathbb{C}}$ in families, necessitating our more subtle gluing definition.

The diagram

$$\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}} \,/\, \mathrm{U}(1)_{\mathrm{Betti}} \leftarrow \mathbb{P}^1_{\mathbb{C}}$$

will be the model for a more general diagram

$$X^{\diamondsuit} \to X^{\diamondsuit} / \,\mathrm{U}(1)_{\mathrm{Betti}} \leftarrow X^{\mathrm{tw}}$$

for any complex manifold $\mathbb{C}$, recovering the first diagram for $X = *$, where $X^{\diamondsuit}$ is analogous to the diamondization construction in p-adic geometry and X^{tw} is a new object we might call the "twistorization." Vector bundles on these stacks should be given respectively by families of vector bundles with two T-connections in a certain sense, generalizing vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$; variations of complex Hodge structures; and variations of complex twistor structures. (In each case Galois descent provides real analogues, provided X descends to $\mathbb{R}$.) In the final section of today's lecture we aim to explain how these stacks are constructed; a more detailed investigation will be the first goal of next week's talk.

In the p-adic situation, for a suitable adic space X the functor $X^{\diamondsuit}$ sends a characteristic p perfectoid space S to the space of untilts $S^{\sharp}$ together with maps $S^{\sharp} \to X$. Thus for example $(\mathrm{Spa}\,\mathbb{Z}_p)^{\diamondsuit}$ classifies all untilts, while $(\mathrm{Spa}\,\mathbb{Q}_p)^{\diamondsuit}$ classifies those in characteristic 0.

In our setting, we don't have a notion of untilts. However, recalling that an untilt $S^{\sharp}$ of S over, say, $\mathbb{Q}_p$ embeds into the Fargues–Fontaine curve as a degree 1 divisor, we do have an analogue here: namely an "untilt" of a test object, here a totally disconnected $\mathbb{C}$-algebra A, should just be an A-point of Div^1, i.e. a degree 1 divisor $Z \subset X_{\mathbb{R},A}$. For simplicity we first work over $\mathbb{C}$: then for a complex manifold X (viewed as a totally disconnected stack), we define $X^{\diamondsuit}$ to be the totally disconnected stack sending A to the space of degree 1 divisors $Z \subset X_{\mathbb{C},A}$ together with a map $Z \to X$.

[1] I think—this formula isn't in Scholze's notes so should be taken with a grain of salt.

Remembering only the divisor Z gives a map $X^\diamondsuit \to \mathrm{Div}^1_{\mathbb{C}}$ for every X. In fact, by inspection $\mathrm{Div}^1_{\mathbb{C}} \simeq (\mathrm{AnSpec}\,\mathbb{C}_{\mathrm{gas}})^\diamondsuit$, and so this is just the diamondization of the projection $X \to \mathrm{AnSpec}\,\mathbb{C}_{\mathrm{gas}}$. When X descends to $\mathbb{R}$, $X^\diamondsuit$ descends to Div^1 and so we can think of diamondization on real manifolds as classifying degree 1 divisors of $X_{\mathbb{R},A}$ over X, with $(\mathrm{AnSpec}\,\mathbb{R}_{\mathrm{gas}})^\diamondsuit = \mathrm{Div}^1$.

Beyond Galois descent, everything descends to $(\mathrm{AnSpec}\,\mathbb{C}_{\mathrm{gas}})/\,\mathrm{O}(2)_{\mathrm{Betti}}$ as above; we again restrict to the $\mathrm{U}(1)_{\mathrm{Betti}}$-action. This gives a stack $X^\diamondsuit/\,\mathrm{U}(1)_{\mathrm{Betti}} \to \mathrm{Div}^1_{\mathbb{C}}/\,\mathrm{U}(1)_{\mathrm{Betti}}$, whose base change along $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}/\,\mathrm{U}(1)_{\mathrm{Betti}}$ recovers $X^\diamondsuit$. This in turn suggests carrying out a similar base change on the other side: we set

$$X^{\mathrm{tw}} = X^\diamondsuit/\,\mathrm{U}(1)_{\mathrm{Betti}} \times_{\mathrm{Div}^1_{\mathbb{C}}/\,\mathrm{U}(1)_{\mathrm{Betti}}} \mathbb{P}^1_{\mathbb{C}},$$

mapping via the second projection to $\mathbb{P}^1_{\mathbb{C}}$. This gives us our diagrams

$$\begin{array}{ccccc} X^\diamondsuit & \longrightarrow & X^\diamondsuit/\,\mathrm{U}(1)_{\mathrm{Betti}} & \longleftarrow & X^{\mathrm{tw}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Div}^1_{\mathbb{C}} & \longrightarrow & \mathrm{Div}^1_{\mathbb{C}}/\,\mathrm{U}(1)_{\mathrm{Betti}} & \longleftarrow & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

as expected, with each square Cartesian. The real version, when X is a real-analytic manifold, is then

$$\begin{array}{ccccc} X^\diamondsuit & \longrightarrow & X^\diamondsuit/\,\mathrm{U}(1)_{\mathrm{Betti}} & \longleftarrow & X^{\mathrm{tw}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Div}^1 & \longrightarrow & \mathrm{Div}^1/\,\mathrm{U}(1)_{\mathrm{Betti}} & \longleftarrow & X_{\mathbb{R}} \end{array}.$$

Note though that the appearance of the twistor $\mathbb{P}^1$ here, while not coincidental, is not the same as the version in families: this does not tell us how to generalize the twistor $\mathbb{P}^1$ to families, but instead how to generalize its transmutative properties, i.e. the relationship between vector bundles on it and twistor structures.

Next time, we will study vector bundles on each of these stacks, and see that we obtain variations of Hodge structure on the middle term (generalizing vector bundles on $\mathrm{Div}^1_{\mathbb{C}}/\,\mathrm{U}(1)_{\mathrm{Betti}}$ and $\mathrm{Div}^1/\,\mathrm{U}(1)_{\mathrm{Betti}}$), variations of twistor structures on the right (generalizing vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ and $X_{\mathbb{R}}$), and a new notion on the left generalizing vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ and Div^1 to the relative setting. For the moment, we can make the following observation about $X^\diamondsuit$ away from 0 and ∞:

Proposition 6. *The projection to the Betti stack yields an isomorphism*

$$X^\diamondsuit \times_{\mathrm{Div}^1_{\mathbb{C}}} \mathbb{C}^\times_{\mathrm{Betti}} \xrightarrow{\sim} (X^\diamondsuit \times_{\mathrm{Div}^1_{\mathbb{C}}} \mathbb{C}^\times_{\mathrm{Betti}})_{\mathrm{Betti}} \simeq X(\mathbb{C})_{\mathrm{Betti}} \times \mathbb{C}^\times_{\mathrm{Betti}}.$$

Proof. Away from 0 and ∞, the relative curve $X_{\mathbb{C},A}$ is just the relative punctured $\mathbb{P}^1_{\mathbb{C}}$, i.e. $\mathbb{G}^{\mathrm{an}}_{\mathrm{m},\mathrm{Cont}(S,\mathbb{C})}$ and so each degree 1 divisor is isomorphic to $\mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{C})$, so the left-hand side is the functor sending A to a point of $\mathbb{C}^\times_{\mathrm{Betti}}(A)$ describing the location of the divisor together with a map $\mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{C}) \to X$, i.e. $X(\mathbb{C})_{\mathrm{Betti}} \times \mathbb{C}^\times_{\mathrm{Betti}}$. □

Next time, we'll study what happens over the missing points. Once we finish our discussion of vector bundles on $\mathrm{Div}^1 = *^\diamond$ and its relative versions $X^\diamond$, we'll introduce the stack Bun_G and study its properties, especially the action of Hecke operators. This puts us in position to sketch the statement of the main geometrized local Langlands conjecture over $\mathbb{R}$.