Nonabelian real Lubin–Tate theory

December 2, 2024

1. The isomorphism of the Lubin–Tate and Drinfeld towers

Last time, we introduced the stack Bun_G together with the Hecke stack

$$\mathrm{Bun}_G \longleftarrow \mathrm{Hck}_\mu \longrightarrow \mathrm{Bun}_G \times \mathrm{Div}^1$$

for each minuscule cocharacter μ, giving rise to Hecke operators by pull-push. Today, we'll specialize to the case $G = \mathrm{GL}_2$ and look at the simplest nontrivial piece of the Hecke correspondence, parametrizing modifications of the trivial GL_2-bundle $\mathcal{O}^2$.

We first recall the situation in the p-adic case, which is closely analogous. The simplest minuscule modification we could ask for is of type $\mu = (1,0)$, which corresponds to an injection $\mathcal{O}^2 \to \mathcal{E}$ which is an isomorphism everywhere except at a fixed point ∞ on the Fargues–Fontaine curve where it has cokernel of length 1, giving a flag, i.e. a point of $\mathrm{Fl}_\mu = \mathbb{P}^1_{\mathbb{C}_p}$. Away from the $\mathbb{Q}_p$-points, this forces $\mathcal{E}$ to in fact be $\mathcal{O}(1/2)$ as it has degree 1 and rank 2, and so we get a morphism

$$\mathcal{M}_{\mathrm{Drinf},\infty} = \{\mathcal{O}^2 \hookrightarrow \mathcal{O}(1/2)\} \to \mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{Q}_p) = \Omega^1,$$

with base Drinfeld's upper half-plane. This is the Drinfeld tower at infinite level, and as the projection can be viewed as forgetting the choice of isomorphism $\mathcal{E} \simeq \mathcal{O}(1/2)$ it is a $\mathrm{Aut}(\mathcal{O}(1/2)) \simeq D^\times$-torsor, where $D/\mathbb{Q}_p$ is the corresponding quaternion algebra. (Over $\mathbb{P}^1(\mathbb{Q}_p)$, the bundle $\mathcal{E}$ is trivial, which for our purposes is a somewhat degenerate case.)

On the other hand, we can also view the objects on the left as modifications of $\mathcal{O}(1/2)$ of type $\mu^{-1} = (0,-1)$, which by forgetting the choice of isomorphism of the modified vector bundle with $\mathcal{O}^2$ gives a projection

$$\{\mathcal{O}^2 \hookrightarrow \mathcal{O}(1/2)\} \to \{\mathcal{O}^2 \hookrightarrow \mathcal{O}(1/2)\}/\operatorname{Aut}(\mathcal{O}^2)).$$

Since $\mathrm{Aut}(\mathcal{O}^2)$ is just $\mathrm{GL}_2(\mathbb{Q}_p)$, this is a $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor, and the base is then the flag variety $\mathrm{Fl}_\mu^{D^\times}$ for $D^\times$ (with respect to μ^-), equivalently its Severi–Brauer variety, which since we work over $\mathbb{C}_p$ we identify with $\mathbb{P}^1_{\mathbb{C}_p}$. This is the Lubin–Tate tower at infinite level

$$\mathcal{M}_{\mathrm{LT},\infty} = \{\mathcal{O}^2 \hookrightarrow \mathcal{O}(1/2)\} \to \mathrm{Fl}_\mu^{D^\times} = \mathbb{P}^1_{\mathbb{C}_p}.$$

The identification of the source here and above is the isomorphism of the Lubin–Tate and Drinfeld towers, and gives a correspondence

$$\mathbb{P}^1_{\mathbb{C}_p} \xleftarrow{\ f\ } \mathcal{M}_{\mathrm{LT},\infty} \simeq \mathcal{M}_{\mathrm{Drinf},\infty} \xrightarrow{\ g\ } \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1_{\mathbb{C}_p}(\mathbb{Q}_p)$$

with commuting actions of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D^\times$ on the source making f a $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor and g a $D^\times$-torsor, with residual $D^\times$-action on the left and $\mathrm{GL}_2(\mathbb{Q}_p)$-action on the right.

Exactly the same logic applies in the archimedean setting, replacing the Fargues–Fontaine curve by the twistor $\mathbb{P}^1$, $\mathrm{GL}_2(\mathbb{Q}_p)$ by $\mathrm{GL}_2(\mathbb{R})$, and D by $\mathbb{H}$, and being careful to take locally analytic groups.[1] We can make this precise as follows.

Theorem 1. *Let $\mathcal{M}$ be the stack sending a totally disconnected $\mathbb{C}$-algebra A to the anima of fiberwise injective maps $i : \mathcal{O}^2_{X_{\mathbb{R},A}} \hookrightarrow \mathcal{O}_{X_{\mathbb{R},A}}(1/2)$.*

(a) The cofiber of i gives a degree 1 divisor on $X_{\mathbb{R},A}$, yielding a morphism $\mathcal{M} \to \mathrm{Div}^1$.

(b) The space of modifications of $\mathcal{O}^2$ of type $(1,0)$ at a degree 1 divisor is $\mathrm{Fl}_\mu^\diamondsuit = \mathbb{P}^{1,\diamondsuit}_{\mathbb{R}}$. Over $(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit \simeq \mathcal{H}^{\pm,\diamondsuit}$, the modified bundle is locally isomorphic to $\mathcal{O}(1/2)$, giving an $\mathbb{H}^{\times,\mathrm{la}}$-torsor

$$g : \mathcal{M} \to (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit.$$[2]

(c) The space of modifications of $\mathcal{O}(1/2)$ of type $(0,-1)$ at a degree 1 divisor is $\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}$, giving a $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-torsor

$$f : \mathcal{M} \to \mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}.$$

In our language from last time, $\mathcal{M}$ is the fiber of Hck_μ over $(\mathcal{O}^2, \mathcal{O}(1/2)) \in \mathrm{Bun}_G \times \mathrm{Bun}_G$, so the map to Div^1 is induced by that from Hck_μ and the actions of $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$ and $\mathbb{H}^{\times,\mathrm{la}}$ are induced from the actions on these points of Bun_G.

One could also view $\mathcal{M}$ as parametrizing pairs of sections of $\mathcal{O}(1/2)$, subject to open injectivity conditions, making $\mathcal{M}$ an open subspace of $\mathcal{BC}(\mathcal{O}(1/2))^2 \simeq \mathbb{A}^{4,\mathrm{an}}$, which in principle can be made explicit so that one could write down all of the maps above concretely.

2. The modular curve

Let $\Gamma \subset \mathrm{GL}_2(\mathbb{Z})$ be a sufficiently small congruence subgroup and consider the modular curve

$$X_\Gamma = \Gamma \backslash (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})).$$

To fully use our tools, it is convenient to pass to its diamondization

$$X_\Gamma^\diamondsuit = \Gamma \backslash (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit.$$

The $\mathbb{H}^{\times,\mathrm{la}}$-torsor $\mathcal{M} \to (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit$ then gives a $\mathbb{H}^{\times,\mathrm{la}}$-torsor

$$\widetilde{X}_\Gamma := \Gamma \backslash \mathcal{M} \to \Gamma \backslash (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit = X_\Gamma^\diamondsuit,$$

where Γ acts on $\mathcal{M}$ via its inclusion into $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$.

[1] Note that in this case the Severi–Brauer variety $\mathrm{Fl}_\mu^{\mathbb{H}^\times}$ is actually the twistor $\mathbb{P}^1$ again; but this is coincidental, bearing no (apparent) relation to its appearance as the archimedean Fargues–Fontaine curve, so we avoid writing it as $X_{\mathbb{R}}$.

[2] Scholze's notes have all of $\mathrm{Fl}_\mu^\diamondsuit$ as the target here, but I'm guessing this is a typo as on the whole space this map isn't surjective.

We can interpret this $\mathbb{H}^{\times,\mathrm{la}}$-torsor as a morphism $X_\Gamma^\diamondsuit \to */\mathbb{H}^{\times,\mathrm{la}}$. The target embeds into $\mathrm{Bun}_{\mathrm{GL}_2}$, and in fact recall we have a map $\mathrm{Bun}_{\mathrm{GL}_2} \to */\mathrm{GL}_2$ via pulling back the GL_2-torsor on the twistor $\mathbb{P}^1$ along the point at infinity, so we can compose these to get a map

$$X_\Gamma^\diamondsuit \to */\mathbb{H}^{\times,\mathrm{la}} \to \mathrm{Bun}_{\mathrm{GL}_2} \to */\mathrm{GL}_2,$$

corresponding to a rank 2 vector bundle V on $X_\Gamma^\diamondsuit$.

On the other hand, X_Γ is equipped with a universal elliptic curve $f_\Gamma : E_\Gamma \to X_\Gamma$, whose first relative cohomology $R^1 f_{\Gamma,*}\mathcal{O}$ is a rank 2 vector bundle on X_Γ, and the same thing holds after diamondization, i.e. $R^1 f_{\Gamma,*}^\diamondsuit \mathcal{O}$ is a rank 2 vector bundle on $X_\Gamma^\diamondsuit$. We claim that these agree:

Theorem 2. *There is an isomorphism $V \simeq R^1 f_{\Gamma,*}^\diamondsuit \mathcal{O}$ of vector bundles on $X_\Gamma^\diamondsuit$.*

There is actually something very surprising about this: the vector bundle V comes from a vector bundle on the curve $X_\mathbb{R}$, while the pushforward of diamond structures has to do with the mirror curve Div^1; and modifications of vector bundles (corresponding to $\mathcal{M}$) are allowed to be over varying points, while the filtrations involved in the variations of Hodge and twistor structures we saw were fixed at the point at infinity.

Proof. Let A be a totally disconnected $\mathbb{C}$-algebra with a map $\mathrm{AnSpec}\, A \to X_\Gamma^\diamondsuit$. This is equivalent to a degree 1 Cartier divisor $Z \subset X_{\mathbb{R},A}$ together with a map $Z \to X_\Gamma$, which corresponds to an elliptic curve $E_Z \to Z$.

In general in this situation where we have an abstract family of twistor $\mathbb{P}^1$'s (here $X_{\mathbb{R},A}$) together with a degree 1 divisor and an object over it, we can construct a version of prismatic cohomology, via the analytic prismatization. This is still in progress in general, though I would like to understand more, but we explain the construction only in the relevant case: $(E_Z/X_{\mathbb{R},A})^{\mathbb{\Delta}}$ is a stack over $X_{\mathbb{R},A}$, defined as follows.

For a test object $\mathrm{AnSpec}\, B \to X_{\mathbb{R},A}$, we define an analytic stack $X'_{\mathbb{R},B}$ which is defined exactly as $X_{\mathbb{R},B}$ except that the fixed point $\mathrm{AnSpec}\,\mathrm{Cont}(S(B),\mathbb{C}) \to X_\mathbb{R} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S(B),\mathbb{C})$ is via the product of $\mathrm{AnSpec}\,\mathrm{Cont}(S(B),\mathbb{C}) \to \mathrm{AnSpec}\, B \to X_{\mathbb{R},A} \to X_\mathbb{R}$ with the identity, rather than via the point at infinity. Similarly to the usual twistor $\mathbb{P}^1$ in families, this can be understood via a universal property: it is the initial family of twistor $\mathbb{P}^1$'s whose fiber at this point admits a map from $\mathrm{AnSpec}\, B$. In particular it follows that the map $\mathrm{AnSpec}\, B \to X_{\mathbb{R},A}$ factors through $X'_{\mathbb{R},B}$.

Now we can define $(E_Z/X_{\mathbb{R},A})^{\mathbb{\Delta}}$ over $X_{\mathbb{R},A}$ to be the stack sending $\mathrm{AnSpec}\, B \to X_{\mathbb{R},A}$ to the anima of maps

$$X'_{\mathbb{R},B} \times_{X_{\mathbb{R},A}} Z \to E_Z$$

over Z, which we can think of as sections of the universal elliptic curve over Z after pullback to $X'_{\mathbb{R},B}$; a priori this is over $X_{\mathbb{C},A}$ but one can check that it descends. If we base change along the canonical map $\mathrm{AnSpec}\, A \to X_{\mathbb{R},A}$, i.e. we require that $\mathrm{AnSpec}\, B \to X_{\mathbb{R},A}$ factors through $\mathrm{AnSpec}\, A$ at the point at infinity, then this is classifying degree 1 divisors $Z \subset X_{\mathbb{R},A}$ factoring through $\mathrm{AnSpec}\, A$ together with maps $Z \to X_\Gamma$ (corresponding to the elliptic curve $E_Z \to Z$) together with a section of the elliptic curve, which is to say a map $Z \to E_\Gamma$

factoring $Z \to X_\Gamma$. Thus we have a Cartesian diagram

$$\begin{array}{ccc} E_\Gamma^\diamondsuit \times_{X_\Gamma^\diamondsuit} \operatorname{AnSpec} A & \longrightarrow & \operatorname{AnSpec} A \\ \downarrow & & \downarrow \\ (E_Z/X_{\mathbb{R},A})^{\mathbbl{\Delta}} & \longrightarrow & X_{\mathbb{R},A} \end{array} .$$

We claim that the first relative cohomology of the bottom map $(E_Z/X_{\mathbb{R},A})^{\mathbbl{\Delta}} \to X_{\mathbb{R},A}$ is a rank 2 vector bundle on $X_{\mathbb{R},A}$ which is locally isomorphic to $\mathcal{O}(1/2)$, so that varying A and the A-point of $X_\Gamma^\diamondsuit$ we get a map $X_\Gamma^\diamondsuit \to \operatorname{Bun}^b_{\mathrm{GL}_2} \simeq */\mathbb{H}^{\times,\mathrm{la}}$. This corresponds to the same $\mathbb{H}^{\times,\mathrm{la}}$-torsor as V; we'll justify this a little more below. Then by the above Cartesian diagram, as A varies the relative cohomology of the bottom map is given by that of the top map, which is $R^1 f^\diamondsuit_{\Gamma,*}\mathcal{O}$ as desired.

To complete the proof, we want to study this prismatic cohomology. Given $\operatorname{AnSpec} A \to X_\Gamma^\diamondsuit$ corresponding to Z and E_Z as above, given $\operatorname{AnSpec} B \to X_{\mathbb{R},A}$ we saw that it factors through $X'_{\mathbb{R},B} \to X_{\mathbb{R},A}$, along which we can pull back Z. Translating the $\operatorname{Cont}(S(B),\mathbb{C})$-point to infinity (which introduces an ambiguity of the stabilizer group $\mathrm{O}(2)_{\mathrm{Betti}}$), this gives a degree 1 divisor $\operatorname{AnSpec} B \to \operatorname{Div}^1 / \mathrm{O}(2)_{\mathrm{Betti}}$; in fact since the resulting divisor maps to Z, this lifts to $\operatorname{AnSpec} B \to Z^\diamondsuit / \mathrm{O}(2)_{\mathrm{Betti}}$. Letting B vary, we get a map $X_{\mathbb{R},A} \to Z^\diamondsuit / \mathrm{O}(2)_{\mathrm{Betti}}$. On the other hand we have a natural map $E_Z^\diamondsuit / \mathrm{O}(2)_{\mathrm{Betti}} \to Z^\diamondsuit / \mathrm{O}(2)_{\mathrm{Betti}}$, and taking the fiber product we get a stack sending B to maps from the pullback of Z to $X'_{\mathbb{R},B}$ to E_Z, which is precisely the definition of $(E_Z/X_{\mathbb{R},A})^{\mathbbl{\Delta}}$, i.e. we have a Cartesian diagram

$$\begin{array}{ccc} (E_Z/X_{\mathbb{R},A})^{\mathbbl{\Delta}} & \longrightarrow & X_{\mathbb{R},A} \\ \downarrow & & \downarrow \\ E_Z^\diamondsuit / \mathrm{O}(2)_{\mathrm{Betti}} & \longrightarrow & Z^\diamondsuit / \mathrm{O}(2)_{\mathrm{Betti}} \end{array} .$$

Therefore it suffices to understand the relative cohomology of the bottom map. We can view this as the Galois descent of $E_Z^\diamondsuit / \mathrm{U}(1)_{\mathrm{Betti}} \to Z^\diamondsuit / \mathrm{U}(1)_{\mathrm{Betti}}$, and we recall that $Y^\diamondsuit / \mathrm{U}(1)_{\mathrm{Betti}}$ classifies variations of Hodge structures on Y, so this is the variation of real Hodge structures cohomology; in particular the first cohomology is a rank 2 vector bundle with determinant the Tate twist in degree 1, and so must be locally $\mathcal{O}(1/2)$. □

We can understand the $\mathbb{H}^{\times,\mathrm{la}}$-torsor on $X_\Gamma^\diamondsuit$ as via the diagram

$$\begin{array}{ccccccc} X_\Gamma^\diamondsuit & \xrightarrow{\quad r \quad} & & \operatorname{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit} / \mathbb{H}^{\times,\mathrm{la}} & \xrightarrow{u} & \operatorname{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}} & \xrightarrow{\mathrm{id}\times i_b} & \operatorname{Div}^1 \times \operatorname{Bun}_{\mathrm{GL}_2} \\ \downarrow{\scriptstyle s} & & & \downarrow{\scriptstyle t} & & & & \\ */\Gamma & \xrightarrow{a} & */\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} & \xrightarrow{i_1} \operatorname{Bun}_{\mathrm{GL}_2} & & & & \end{array}$$

where the rectangle is Cartesian and $\operatorname{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit} / \mathbb{H}^{\times,\mathrm{la}} \simeq \mathcal{M}/(\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \times \mathbb{H}^{\times,\mathrm{la}})$ can be thought of as the piece of the Hecke stack after quotienting by the relevant automorphism groups, so parametrizing modifications $\mathcal{E} \hookrightarrow \mathcal{E}'$ where $\mathcal{E}$ is isomorphic to $\mathcal{O}^2$ and $\mathcal{E}'$ to $\mathcal{O}(1/2)$ but without requiring the data of these isomorphisms, and the maps to $\mathrm{Bun}_{\mathrm{GL}_2}$ and $\mathrm{Div}^1 \times \mathrm{Bun}_{\mathrm{GL}_2}$ are the resulting Hecke correspondence. In particular the canonical $\mathbb{H}^{\times,\mathrm{la}}$-torsor $\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}$ over $\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}}$ pulls back to $(\mathcal{M}/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}) \times_{*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}} */\Gamma \simeq \Gamma\backslash\mathcal{M} = \widetilde{X}_\Gamma^\diamondsuit$. This Cartesian diagram is the archimedean analogue of Zhang's (in general conjectural) Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sh}_{K^p}^\diamondsuit & \longrightarrow & \mathrm{Gr}_\mu \\ \downarrow & & \downarrow \\ \mathrm{Igs}_{K^p} & \longrightarrow & \mathrm{Bun}_G \end{array},$$

which I hope to elaborate on next time.

In particular, the fact that the diagram commutes implies that the map $X_\Gamma^\diamondsuit \to */\mathbb{H}^{\times,\mathrm{la}}$ which we constructed over the course of the proof corresponds to the torsor arising as the pullback of the torsor $\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit} \to \mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}}$, equivalently a modification of the $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-torsor corresponding to $X_\Gamma^\diamondsuit \to */\Gamma \to */\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$ parametrized by the flag variety, which is to say $\mathcal{M}/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$. Its pullback to $X_\Gamma^\diamondsuit$ is therefore just $\Gamma\backslash\mathcal{M} = \widetilde{X}_\Gamma$, the torsor corresponding to V above.

The Cartesian diagram also implies a version of Matsushima's formula relating the cohomology of X_Γ to automorphic forms. Let $\pi_\Gamma = a_!(1)$, which we think of as a version of the space of cusp forms of level Γ. We write $p : \mathrm{Div}^1 \times \mathrm{Bun}_{\mathrm{GL}_2} \to \mathrm{Div}^1$ for the projection, and $q : X_\Gamma \to *$ for the structure map so that we have the induced map $q^\diamondsuit : X_\Gamma^\diamondsuit \to \mathrm{Div}^1$. For each algebraic representation ρ of GL_2, pulling back along the canonical map

$$X_\Gamma^\diamondsuit \to */\mathrm{GL}_2$$

as above we get a vector bundle V_ρ on $X_\Gamma^\diamondsuit$; since the above map factors through $*/\mathbb{H}^{\times,\mathrm{la}}$ we also get a locally analytic representation of $\mathbb{H}^\times$, which by an abuse of notation we also denote by ρ.

Taking the relative cohomology

$$R\Gamma_c(X_\Gamma^\diamondsuit, V_\rho) := q_!^\diamondsuit V_\rho$$

gives an object of $D(\mathrm{Div}^1)$. On the other hand, the Hecke correspondence $T_\mu : D(\mathrm{Bun}_{\mathrm{GL}_2}) \to D(\mathrm{Div}^1 \times \mathrm{Bun}_{\mathrm{GL}_2})$ lets us send the object π_Γ of $D(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})$, after embedding into $D(\mathrm{Bun}_{\mathrm{GL}_2})$, to an object of $D(\mathrm{Div}^1 \times \mathrm{Bun}_{\mathrm{GL}_2})$. Via the diagram above, we can view this as

$$T_\mu(\pi_\Gamma) = (\mathrm{id} \times i_b)_! u_! t^* i_{1!} a_!(1).$$

By proper base change along the Cartesian square,

$$t^* i_{1!} a_!(1) \simeq r_! s^*(1) \simeq r_!(1)$$

so this is the !-pushforward along the composite map $X_\Gamma^\diamondsuit \to \mathrm{Div}^1 \times \mathrm{Bun}_G$. To get something just on Div^1, we'll take the pushforward along p.

We were previously interested in the cohomology just of V_ρ, so this pushforward is not quite right: we modify it by taking the "$\rho^\vee$-isotypic piece," i.e.

$$\begin{aligned} p_!(T_\mu(\pi_\Gamma) \otimes (1 \boxtimes i_{b!}\rho)) &\simeq p_!((1 \boxtimes i_{b!}\rho) \otimes (\mathrm{id} \times i_b)_! u_! r_! 1) \\ &\simeq p_!(\mathrm{id} \times i_b)_!((1 \boxtimes \rho) \otimes u_! r_! 1) \\ &\simeq p_!(\mathrm{id} \times i_b)_! u_! r_!(r^* u^*(1 \boxtimes \rho)) \\ &\simeq q_!^\diamondsuit V_\rho. \end{aligned}$$

Abbreviating the left-hand side, we arrive at the following theorem:

Theorem 3. *There is an isomorphism*

$$R\Gamma_c(X_\Gamma^\diamondsuit, V_\rho) \simeq p_!(T_\mu(\pi_\Gamma) \otimes \rho)$$

in $D(\mathrm{Div}^1)$.

3. Jacquet–Langlands/local Langlands

Classically, the main reason to care about the Lubin–Tate or Drinfeld towers is that they give spaces with compatible actions of the Weil group of $\mathbb{Q}_p$ and of $\mathrm{GL}_2(\mathbb{Q}_p)$ or the quaternion group $D^\times$, which in some sense realizes the local Langlands correspondence in a way similar to how the cohomology of modular curves realizes the global Langlands correspondence for GL_2. From this point of view, the isomorphism of the Lubin–Tate and Drinfeld towers can be viewed as a statement about the relationship between local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$ and for $D^\times$. Explicitly, if π is a suitable irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_\ell$, via the $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor

$$\mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^1_{\mathbb{C}_p}$$

we can construct a sheaf $\mathcal{F}_\pi$ on $\mathbb{P}^1_{\mathbb{C}_p}$ whose stalks are π, which is equipped with $D^\times$-equivariance and a Weil descent datum giving actions of $W_{\mathbb{Q}_p} \times D^\times$ on $H^*(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$. (Explicitly, $\mathcal{F}_\pi$ is the pullback of π, viewed as a vector bundle on $*/\mathrm{GL}_2(\mathbb{Q}_p)$, along the map $\mathbb{P}^1_{\mathbb{C}_p} \to */\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to the torsor $\mathcal{M}_{\mathrm{LT},\infty}$.)

On the other hand, associated to π we have the Jacquet–Langlands correspondence $\mathrm{JL}(\pi)$, an irreducible $D^\times$-representation, and the local Langlands correspondence $\mathrm{LLC}(\pi)$, a two-dimensional $W_{\mathbb{Q}_p}$-representation. A theorem of Deligne states that together these give the above construction, up to a Tate twist:

Theorem 4. *If π is a discrete series irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then $H^*(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$ is concentrated in degree 1, where it is isomorphic to $\mathrm{JL}(\pi) \otimes \mathrm{LLC}(\pi)(-1/2)$.*

Our goal is to develop and prove a real analogue of this result, giving a cohomological realization of the local Langlands and Jacquet–Langlands correspondences.

In fact, one can show an analogous result for GL_n in place of GL_2. However for our real analogue we will need to restrict to groups with Shimura varieties, so we restrict ourselves to GL_2.

Using the machinery of the previous sections, it is not hard to see how to adapt the setup to the archimedean place: the $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor $\mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^1_{\mathbb{C}_p}$ is replaced with the

$\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-torsor $\mathcal{M} \to \mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}$. Via the isomorphism of the two towers we have a composite projection

$$\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}} \simeq (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit / \mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \to */\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$$

so that any $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-representation π pulls back to a sheaf $\mathcal{F}_\pi$ on $\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}}$, or equivalently to a $\mathbb{H}^\times$-equivariant sheaf which we again denote by $\mathcal{F}_\pi$ on $\mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}$, with stalks isomorphic to π. In our diagram above, we could think of this as

$$\mathcal{F}_\pi = t^* i_{1!}\pi.$$

Now recall we have the projection

$$u : \mathrm{Fl}_\mu^{\mathbb{H}^\times,\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}} \to \mathrm{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}},$$

which is cohomologically smooth and proper. We are interested in

$$Ru_*\mathcal{F}_\pi \in D(\mathrm{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}}).$$

Via our interpretation of $\mathcal{F}_\pi$ as a pullback, this is essentially the simplest Hecke operator T_μ applied to π, after restricting to the $*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$- and $*/\mathbb{H}^{\times,\mathrm{la}}$-loci in $\mathrm{Bun}_{\mathrm{GL}_2}$. We define this operation to be the Jacquet–Langlands/local Langlands functor

$$\pi \mapsto \mathrm{JLL}(\pi) = Ru_*\mathcal{F}_\pi.$$

This is our version of the assignment

$$\pi \mapsto H^*(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$$

in the p-adic case; here we explicitly retain the quaternion algebra action and the Weil group action (via Div^1).

Theorem 5. *If π is a discrete series irreducible representation of $\mathrm{GL}_2(\mathbb{R})$, then*

$$\mathrm{JLL}(\pi) \simeq \mathrm{JL}(\pi) \otimes \mathrm{LLC}(\pi)[-1](-1/2).$$

Here $\mathrm{LLC}(\pi)$ is a rank 2 vector bundle on Div^1 corresponding to the L-parameter of π, $\mathrm{JL}(\pi)$ is an irreducible finite-dimensional $\mathbb{H}^\times$-representation, and the twist $(-1/2)$ denotes a positive square root of the norm character $|\cdot|$ on $W_\mathbb{R}$.

We can think of this theorem as giving a cohomological realization of the local Langlands correspondence for GL_2 on the archimedean Lubin–Tate tower, analogous to the p-adic case; more generally a similar construction should be possible for any group admitting a Shimura variety. The main goal of the rest of today's talk will be to prove Theorem 5.

4. Infinitesimal character

Similar to how we approached Beilinson–Bernstein localization, we first note that each of $D(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})$ and $D(*/\mathbb{H}^{\times,\mathrm{la}})$ is naturally linear over the Harish–Chandra center $U(\mathfrak{h})^W$ where $\mathfrak{h} \subset \mathfrak{gl}_2$ is the Cartan subalgebra. The first question is then whether the functor JLL preserves this action, i.e. preserves infinitesimal characters. We claim the answer is yes:

Proposition 6. *The functor* JLL *is naturally* $U(\mathfrak{h})^W$*-linear.*

The most difficult part is when the modification in the Hecke operator occurs at ∞. Away from ∞, we can make a more general statement:

Proposition 7. *For any reductive real group G with Cartan algebra $\mathfrak{h}$, the category $D(\mathrm{Bun}_G)$ is naturally linear over $U(\mathfrak{h})^W$, and away from ∞ the Hecke operators*

$$T_\mu|_{\mathrm{Div}^1\setminus\{\infty\}} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times \mathrm{Div}^1)$$

are naturally $U(\mathfrak{h})^W$*-linear.*

Proof. Recall that we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Bun}_G & \longrightarrow & (\mathrm{Bun}_G)_{\mathrm{Betti}} \\ \downarrow & & \downarrow \\ */G_{\mathbb{C}}^{\mathrm{an}} & \longrightarrow & */G(\mathbb{C})_{\mathrm{Betti}} \end{array}$$

so in particular Bun_G descends along the map $*/G_{\mathbb{C}}^{\mathrm{an}} \to */G(\mathbb{C})_{\mathrm{Betti}}$, and away from ∞ the Hecke operators descend as well. Thus it suffices to prove the lemma below. □

Lemma 8. *For any analytic stack $X \to */G(\mathbb{C})_{\mathrm{Betti}}$, let $\widetilde{X} = X \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}}$. Then the category $D(\widetilde{})$ is naturally $U(\mathfrak{h})^W$-linear, compatibly with all operations.*

Here "operations" should I think be understood as integral transforms against some kernel along self-correspondences. In particular in the case of the proposition we have compatibility with the Hecke operators.

Proof. Via the correspondence

$$\begin{array}{ccc} & X \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}} \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}} & \\ \swarrow & & \searrow \\ X \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}} & & X \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}} \end{array}$$

using the two possible projections, we get an action of $D(*/G_{\mathbb{C}}^{\mathrm{an}} \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}})$ on $D(\widetilde{X})$ by kernels, compatibly with all operations. The kernel sheaf $\Delta_!1$ acts as the identity for

$$\Delta : */G_{\mathbb{C}}^{\mathrm{an}} \to */G_{\mathbb{C}}^{\mathrm{an}} \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}}$$

the diagonal, so $D(\widetilde{X})$ is $\mathrm{End}(\Delta_!1)$-linear as desired. But the infinitesimal character gives an action of $U(\mathfrak{h})^W$ on $\Delta_!1$, so the composition is the desired action. □

Proof of Proposition 6. The idea is to spread out the linearity over $\mathrm{Div}^1\setminus\{\infty\}$ from Proposition 7 to all of Div^1.

Consider the correspondence

$$\begin{array}{ccc} & \mathrm{Fl}_{\mu}^{\mathbb{H}^{\times},\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}} & \\ {\scriptstyle t'}\swarrow & & \searrow{\scriptstyle u} \\ */\operatorname{GL}_2(\mathbb{R})^{\mathrm{la}} & & \operatorname{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}} \end{array}$$

where t' is via the isomorphism of the two towers as above. We have $\operatorname{JLL}(\pi) = Ru_* t'^* \pi$. We could also however view JLL as the integral transform along

$$\begin{array}{ccc} & */\operatorname{GL}_2(\mathbb{R})^{\mathrm{la}} \times \operatorname{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}} & \\ {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ */\operatorname{GL}_2(\mathbb{R})^{\mathrm{la}} & & \operatorname{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}} \end{array}$$

with respect to the kernel $K = (t' \times u)_! 1$, as

$$R\,\mathrm{pr}_{2*}(K \otimes \mathrm{pr}_1^* \pi) = R\,\mathrm{pr}_{2*}((t' \times u)_! 1 \otimes \mathrm{pr}_1^* \pi) \simeq R\,\mathrm{pr}_{2*}(t' \times u)_!(t' \times u)^* \mathrm{pr}_1^* \pi \simeq Ru_* t'^* \pi.$$

Via the contributions of each classifying stack we get two $U(\mathfrak{h})^W$-actions on K, and the $U(\mathfrak{h})^W$-linearity of the integral transform against K is then the identification of these two actions.

Recall that for a maximal compact subgroup $K \subset G$, the inclusion

$$*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^{\dagger} \to */G^{\mathrm{la}}$$

classifies the realization of locally analytic representations as (analytic) $(\mathfrak{g}, K)$-modules. Since the $U(\mathfrak{h})^W$-actions depend only on the Lie algebra actions, we can study the actions on K after pullback to

$$*/(K^{\mathrm{la}} \subset \operatorname{GL}_2(\mathbb{R})^{\mathrm{la}})^{\dagger} \times \operatorname{Div}^1 \times */(K_{\mathbb{H}}^{\mathrm{la}} \subset \mathbb{H}^{\times,\mathrm{la}})^{\dagger}$$

where $K \subset \operatorname{GL}_2(\mathbb{R})$ and $K_{\mathbb{H}} \subset \mathbb{H}^{\times}$ are maximal compact subgroups. By analytic Riemann–Hilbert and base change along the resulting Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}(\mathbb{C})/K \times K_{\mathbb{H}} & \longrightarrow & \mathrm{Fl}_{\mu}^{\mathbb{H}^{\times},\diamondsuit}/\mathbb{H}^{\times,\mathrm{la}} \\ \downarrow & & \downarrow \\ */(K^{\mathrm{la}} \subset \operatorname{GL}_2(\mathbb{R})^{\mathrm{la}})^{\dagger} \times \operatorname{Div}^1 \times */(K_{\mathbb{H}}^{\mathrm{la}} \subset \mathbb{H}^{\times,\mathrm{la}})^{\dagger} & \longrightarrow & */\operatorname{GL}_2(\mathbb{R})^{\mathrm{la}} \times \operatorname{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}} \end{array}$$

we can understand this pullback as the relative compactly supported cohomology of a sheaf on $\mathcal{M}(\mathbb{C})/K \times K_{\mathbb{H}}$. Thus we want to show that for any $K \times K_{\mathbb{H}}$-invariant compact subsets $Z \subset \mathcal{M}(\mathbb{C})$, the $*$-pushforward of (the restriction of) this sheaf along

$$(Z \subset \mathcal{M})^{\dagger}/(K^{\mathrm{la}} \subset \operatorname{GL}_2(\mathbb{R})^{\mathrm{la}})^{\dagger} \times (K_{\mathbb{H}}^{\mathrm{la}} \subset \mathbb{H}^{\times,\mathrm{la}})^{\dagger} \to */(K^{\mathrm{la}} \subset \operatorname{GL}_2(\mathbb{R})^{\mathrm{la}})^{\dagger} \times \operatorname{Div}^1 \times */(K_{\mathbb{H}}^{\mathrm{la}} \subset \mathbb{H}^{\times,\mathrm{la}})^{\dagger}$$

the two $U(\mathfrak{h})^W$-actions agree.

Since this is the pushforward of a quasicoherent sheaf along the inclusion of a compact subset, it is concentrated in degree 0, and since it is a continuous condition it can be checked on the dense open subset

$$*/(K^{\mathrm{la}} \subset \mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})^{\dagger} \times (\mathrm{Div}^1 \setminus \{\infty\}) \times */(K_{\mathbb{H}}^{\mathrm{la}} \subset \mathbb{H}^{\times,\mathrm{la}})^{\dagger}.$$

But here it follows from Proposition 7. □

5. Proof via local-global compatibility

We can now give a proof of Theorem 5 via local-global compatibility. Assume for simplicity that π has trivial central character; otherwise we can fix a central character and make essentially the same argument. Let V_λ be the finite-dimensional representation of GL_2 with highest weight λ with the same infinitesimal character as π; by Proposition 6, $\mathrm{JLL}(\pi)$ also has the infinitesimal character of V_λ. One can check in general that Hecke operators preserve central characters, so $\mathrm{JLL}(\pi)$ has trivial central character and so descends to the compact group $(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}$. Since it is compact, the representations of $\mathbb{H}^\times/\mathbb{R}^\times$ with infinitesimal character λ are generated by V_λ viewed (as above) as a representation of $(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}$, so it suffices to compute the V_λ-isotypic component of $\mathrm{JLL}(\pi)$ as an object of $D(*/(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}} \times \mathrm{Div}^1)$.

The idea is then to globalize π and use Theorem 3. Let $D/\mathbb{Q}$ be a quaternion algebra split at ∞, and set $G = D^\times/\mathbb{G}_{\mathrm{m}}$ with $\Gamma \subset G(\mathbb{Q}) \subset \mathrm{PGL}_2(\mathbb{R})$ a congruence subgroup. We write $\pi_\Gamma = C^\omega(\mathrm{PGL}_2(\mathbb{R})/\Gamma)$ for the space of real-analytic automorphic forms of level Γ, and $\pi_{\Gamma,\lambda}$ its localization at infinitesimal character λ. Since this is over the bounded part of $Z(U(\mathfrak{g})) \simeq U(\mathfrak{h})^W$, we can equivalently view it as an admissible $(\mathfrak{pgl}_2, K)$-module, which (for suitable Γ) contains (the $(\mathfrak{pgl}_2, K)$-module corresponding to) π, so we can find some global cuspidal automorphic representation with multiplicity 1 and component at infinity given by π.

Let

$$X_\Gamma = \Gamma \backslash (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})),$$

which is defined over $\mathbb{R}$. The relative cohomology $R\Gamma_c(X_\Gamma^\diamondsuit, V_\lambda)$ is isomorphic by Theorem 3, in our more recent language, to the V_λ-isotypic component of $\mathrm{JLL}(\pi_\Gamma)$, i.e. $\mathrm{JLL}(\pi_{\Gamma,\lambda})$, compatibly with Hecke operators. After taking Hecke eigenspaces, this gives an isomorphism between $\mathrm{JLL}(\pi)$ and the corresponding eigenspace in $R\Gamma_c(X_\Gamma^\diamondsuit, V_\lambda)$.

On the other hand, V_λ is a vector bundle on $X_\Gamma^\diamondsuit$, which we know we can think of as a "variation of $\diamondsuit$-structures," some generalization of a variation of Hodge structures, and so so is its cohomology in each degree; one can compute that the Hecke eigenspace (cutting out the $\mathrm{LLC}(\pi)$-component) has dimension 2 and write down the Hodge numbers, so the only remaining thing is to identify the character of the cohomology in degree 1 with $\mathrm{LLC}(\pi)(-1/2)$. For π a discrete series, its L-parameter as a rank 2 representation of $W_\mathbb{R}$ is explicit, and so the claim follows from the following lemma.

Lemma 9. *Let V be a rank 2 vector bundle on Div^1 whose restriction to $\mathrm{Div}^1_\mathbb{C}$ has trivial monodromy and corresponds to a $\mathbb{C}$-Hodge structure of type $((p_1, q_1), (p_2, q_2))$ with $p_1 \neq q_1$.*

Then $p_2 = q_1$ and $p_1 = q_2$, the vector bundle V is irreducible, and is given as the pullback along

$$\mathrm{Div}^1 \to */W_{\mathbb{R}}^{\mathrm{la}}$$

of the locally analytic $W_{\mathbb{R}}$-representation

$$\mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}}(z \mapsto z^{p_1}\overline{z}^{q_1}).$$

Proof. The conditions on V together with our previous descriptions of vector bundles on Div^1 and $\mathrm{Div}^1_{\mathbb{C}}$ ensure that V must be the pushforward of the line bundle on $\mathrm{Div}^1_{\mathbb{C}}$ corresponding to the $\mathbb{C}$-Hodge structure of type (p_1, q_1), from which the rest of the description follows. □

6. Local proof

The above proof is not totally satisfying, in that most of the theory as well as the statement of Theorem 5 is purely local but the proof is not. We sketch a purely local proof.

The functor JLL is a composite of a cohomologically smooth proper pushforward and a cohomologically smooth pullback, and hence has good formal properties; in particular one can write down its left adjoint

$$D(*/\mathbb{H}^{\times,\mathrm{la}}) \to D(*/\,\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \times \mathrm{Div}^1)$$

(shifting the Div^1 around) via pull-push along essentially the same correspondence, up to shift and twist, again using the isomorphism of the two towers. For example the trivial representation of $\mathbb{H}^{\times,\mathrm{la}}$ corresponds to the !-pushforward of the structure sheaf along

$$(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamond \to \mathrm{Div}^1$$

with the residual $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-action. Away from ∞, this is the compactly supported cohomology of $\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}) \simeq \mathcal{H}^\pm$, which by Poincaré duality is a 2-dimensional vector space concentrated in degree 2, with the natural action of $\mathrm{GL}_2(\mathbb{R})$; at ∞, it is the compactly supported Hodge cohomology. By analytic Beilinson–Bernstein (and the Matsuki correspondence and its consequences), $R\Gamma_c(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}), \mathcal{O})$ is given by the discrete series representation π with trivial infinitesimal character, while $R\Gamma_c(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}), \Omega^1)$ contains (via the map $\nabla : \mathcal{O} \to \Omega^1$) the discrete series π as a subrepresentation, with cokernel the two-dimensional representation described above. We can perform similar calculations for other representations ρ of $\mathbb{H}^\times$; as in the previous section it suffices to describe the ρ-isotypic components.

One can see in this way that $\mathrm{JLL}(\pi)$ is the tensor product of some ρ (understood as $\mathrm{JL}(\pi)$) with a rank 2 vector bundle on Div^1, which we want to identify with $\mathrm{LLC}(\pi)$. As with Lemma 9, it suffices to understand it after pullback to $\mathrm{Div}^1_{\mathbb{C}}$, where it amounts to understanding the $\mathbb{C}^\times$-action; a suitable description should follow from unraveling the computations.