Analytic Beilinson–Bernstein localization

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The goal of today's talk is to use our geometric perspective on real representation theory to understand Beilinson–Bernstein localization, which relates $\mathfrak{g}$-modules to D-modules on the flag variety. Last time, we related $(\mathfrak{g}, K)$-modules to locally analytic G-representations, which suggests that for suitable versions of Beilinson–Bernstein we should be able to relate locally analytic G-representations to something like D-modules on the flag variety. The right relation is given by the Matsuki correspondence, which can be viewed as combining with Beilinson–Bernstein to give the previous results on $(\mathfrak{g}, K)$-modules and locally analytic G-representations.

1. Algebraic Beilinson–Bernstein

Let Fl be the flag variety of G, parametrizing Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$; we often implicitly fix a representative such that $\mathrm{Fl} \simeq G/B$. More precisely $\mathfrak{g}$ integrates to a formal group $\widehat{G}$, which we can think of as $(1 \subset G)^\wedge$, and the subalgebra $\mathfrak{b}$ integrates to $\widehat{B} \subset \widehat{G}$. The quotient $*/\widehat{B} \to */\widehat{G}$ can be promoted to a universal map $\mathrm{Fl}/\widehat{G} \to */\widehat{G}$ independent of a choice of $\widehat{B}$.

For each $\mathfrak{b}$, we have a Cartan quotient $\mathfrak{b} \to \mathfrak{t}$, which lifts to a (constant) sheaf of commutative Lie algebras on Fl giving the "universal Cartan" $\mathfrak{h}$, which in turn integrates to a formal group $\widehat{H}$. Analogously to the map $*/\widehat{B} \to */\widehat{G}$, we hope that the map $*/\widehat{B} \to */\widehat{T}$ lifts to a universal version, which should be something like $\mathrm{Fl}/\widehat{G} \to \mathrm{Fl}/(\widehat{G}/\widehat{U})$ for a (formal) unipotent subgroup $\widehat{U} \subset \widehat{B}$. To make sense of this, we need the following lemma.

Lemma 1. *The action map*

$$\widehat{G} \times \mathrm{Fl} \to \mathrm{Fl}$$

factors through $(\widehat{G} \times \mathrm{Fl})/\widehat{U} \to \mathrm{Fl}$*, which together with the projection gives a groupoid* $(\widehat{G} \times \mathrm{Fl})/\widehat{U} \rightrightarrows \mathrm{Fl}$ *under* $\widehat{G} \times \mathrm{Fl} \rightrightarrows \mathrm{Fl}$*. In fact the same is true after replacing* $\widehat{U}$ *by* $\widehat{B}$*.*

The proof is just the observation that $\widehat{U}$ acts trivially on the space of Borel subalgebras, and then checking that the groupoid structure maps descend. The result is in fact more general, applying to any group G acting on a space X of subgroups $B \subset G$ by conjugation with respect to a characteristic subgroup U.

We denote the resulting stack quotients by $\mathrm{Fl}/(\widehat{G}/\widehat{U})$ and $\mathrm{Fl}/(\widehat{G}/\widehat{B})$ respectively.

Lemma 2. *The map*

$$\mathrm{Fl}/(\widehat{G}/\widehat{U}) \to \mathrm{Fl}/(\widehat{G}/\widehat{B})$$

is a gerbe for $\widehat{H}$*, and there is a unique isomorphism* $\mathrm{Fl}/(\widehat{G}/\widehat{B}) \simeq \mathrm{Fl}_{dR}$ *under* Fl*.*

Proof. The first statement is almost by definition: the fibers are classifying stacks for the universal Cartan (i.e. the formal group of the universal Cartan algebra).

For the second, we know that $\mathrm{Fl}_{\mathrm{dR}}$ is the quotient of $\mathrm{Fl} \times \mathrm{Fl}$ by the formal completion of the diagonal. We claim that the product of the action and projection maps $(\widehat{G} \times \mathrm{Fl})/\widehat{B} \to$

$\mathrm{Fl} \times \mathrm{Fl}$ is injective with image given by the formal completion of the diagonal, so that taking quotients gives the identification as claimed. Since $\widehat{G}$ and $\widehat{B}$ are formal, the map factors through the formal completion of the diagonal, where it is a map of formally smooth formal schemes, an isomorphism on reduced subschemes, and an isomorphism on the first infinitesimal neighborhood (since on the Lie algebras), hence an isomorphism. □

In particular, the pullback $D(\mathrm{Fl}/(\widehat{G}/\widehat{U})) \to D(\mathrm{Fl}/(\widehat{G}/\widehat{B})) \simeq D(\mathrm{Fl}_{\mathrm{dR}})$ is along this $\widehat{H}$-gerbe and so for any character $\chi : \widehat{H} \to \mathbb{G}_m$ we get an isomorphism between the category of χ-twisted D-modules on Fl and the χ-equivariant component of $D(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$, i.e. $D(\mathrm{Fl}/(\widehat{G}/\widehat{U})) \otimes_{U(\mathfrak{h}),\chi} \mathbb{C}$. We refer to this as restriction along the character χ; a particularly important case is that of the trivial character, where we just have usual D-modules.

We are interested in the correspondence

$$*/\widehat{G} \xleftarrow{a} \mathrm{Fl}/\widehat{G} \xrightarrow{b} \mathrm{Fl}/(\widehat{G}/\widehat{U}).$$

We note that a is cohomologically smooth since Fl is, and b is cohomologically smooth and further has the property that the structure sheaf is b-proper with invertible dual (as we've seen before for similar quotient maps; observing that the fibers are $*/\widehat{U}$ we can reduce to the case $*/G^\wedge$ that we've encountered in the past).

In the spirit of Beilinson–Bernstein localization, we'd like to study the operation of pull-pushing along this correspondence. The derived categories of the source and target have different properties and so this won't give an equivalence directly: for example $D(*/\widehat{G})$ is linear over $Z(U(\mathfrak{g}))$, while $D(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$ is linear over $U(\mathfrak{h})$. It turns out that in a suitable sense this is the only obstruction, and we can use the relationships between these two actions to describe the pull-push a_*b^*: we have a map $Z(U(\mathfrak{g})) \to U(\mathfrak{h})$ identifying the source with the W-invariants in the target, so we might hope that the restriction of the $U(\mathfrak{h})$-action to the W-invariant part is compatible with the $Z(U(\mathfrak{g}))$-action, so that a_*b^* would lift to a functor

$$D(\mathrm{Fl}/(\widehat{G}/\widehat{U})) \to D(*/\widehat{G}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}).$$

Indeed this is the case: it suffices to look at the object $(a,b)_!\mathcal{O}_{\mathrm{Fl}/\widehat{G}}$ in $D(*/\widehat{G} \times \mathrm{Fl}/(\widehat{G}/\widehat{U}))$, a category linear over $Z(U(\mathfrak{g})) \otimes U(\mathfrak{h})$. The pullback of this sheaf to the flag variety has stalks given by the compactly supported cohomology of $\widehat{G}/\widehat{U}$, and one can check explicitly that the $Z(U(\mathfrak{g}))$- and $U(\mathfrak{h})$-actions agree along the stated map.

Since b is cohomologically smooth, a_*b^* (and thus its $U(\mathfrak{h})$-augmentation above) admits a left adjoint $b_\sharp a^*$.

Theorem 3 (Algebraic Beilinson–Bernstein)**.** *The functor*

$$b_\sharp a^* : D(*/\widehat{G}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$$

is fully faithful, and its right adjoint a_*b^* is the Verdier quotient by sheaves whose pullbacks to Fl *have trivial cohomology.*

Restricting to regular weights, both functors become equivalences, and restricting to weakly dominant weights both functors become t-exact.

The proof seems quite interesting, but is essentially classical and so we omit it.

In particular for the trivial character we get an equivalence $D(\mathrm{Fl}/(\widehat{G}/\widehat{B})) \simeq D(*/\widehat{G})^{\chi=1}$, which by Lemma 2 gives a further equivalence with $D(\mathrm{Fl}_{\mathrm{dR}}) \simeq D\text{-}\mathbf{Mod}(\mathrm{Fl})$, i.e. $\mathfrak{g}$-modules with trivial infinitesimal character are equivalent to D-modules on the flag variety. This is the essence of Beilinson–Bernstein which we want to extend to the analytic setting. Notably, perfect complexes on each side translate to infinite-dimensional Lie algebra representations (e.g. the regular representation for pushforward of the structure sheaf from a point) but relatively "finite" (regular holonomic) D-modules.

2. Analytic Beilinson–Bernstein

We can think of the formal group $\widehat{G}$ as $(1 \subset G)^\wedge$, and replace it throughout by the analytic version $(1 \subset G)^\dagger$, which we similarly abbreviate to $G^\dagger$, and likewise for the other groups in question. This gives an analytic correspondence

$$*/G^\dagger \xleftarrow{a^\dagger} \mathrm{Fl}/G^\dagger \xrightarrow{b^\dagger} \mathrm{Fl}/(G^\dagger/U^\dagger)$$

with similar geometric properties, a gerbe

$$\mathrm{Fl}/(G^\dagger/U^\dagger) \to \mathrm{Fl}/(G^\dagger/B^\dagger)$$

for $H^\dagger$, and an identification

$$\mathrm{Fl}/(G^\dagger/B^\dagger) \simeq \mathrm{Fl}_{\mathrm{dR}}^{\mathrm{an}} \simeq \mathrm{Fl}_{\mathrm{Betti}}$$

using the analytic Riemann–Hilbert correspondence. The maps from formal to overconvergent completions give rise to a commutative diagram

$$\begin{array}{ccccc} */\widehat{G} & \xleftarrow{a} & \mathrm{Fl}/\widehat{G} & \xrightarrow{b} & \mathrm{Fl}/(\widehat{G}/\widehat{U}) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} \\ */G^\dagger & \xleftarrow{a^\dagger} & \mathrm{Fl}/G^\dagger & \xrightarrow{b^\dagger} & \mathrm{Fl}/(G^\dagger/U^\dagger) \end{array},$$

with the left square Cartesian; from what we've seen in the past it's straightforward to see that each of p_i^* is fully faithful, and by proper base change $p_1^* a_*^\dagger b^{\dagger *} \simeq a_* p_2^* b^{\dagger *} \simeq a_* b^* p_3^*$. The analogue of Theorem 3 is then the following:

Theorem 4. *The functor*

$$b_\sharp^\dagger a^{\dagger *} : D(*/G^\dagger) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D(\mathrm{Fl}/(G^\dagger/U^\dagger))$$

is fully faithful, and its right adjoint $a_*^\dagger b^{\dagger *}$ is the Verdier quotient by sheaves whose pullbacks to Fl *have trivial cohomology. Restricting to regular weights, both functors become equivalences.*

A statement on t-exactness should hold as well, but the t-structures involved are strange and so we omit it.

Proof. It suffices to show that $a^\dagger_* b^{\dagger *}$ and $b^\dagger_\sharp a^{\dagger *}$ pull back along the p_i to the algebraic versions $a_* b^*$ and $b_\sharp a^*$ respectively, since these give fully faithful embeddings into categories on which we already know the analogous statements by Theorem 3. For $a^\dagger_* b^{\dagger *}$ we checked this above by proper base change. In the other direction, we need to see that the restriction of $b_\sharp a^*$ to $D(*/G^\dagger)$ (along p_1^*) lands in the essential image of $D(\mathrm{Fl}/(G^\dagger/U^\dagger))$ (along p_3^*). The fibers of b are copies of $\widehat{U}$ so this can be thought of as the homology of $\widehat{U}$ with respect to representations given by restriction from $\widehat{G}$. Restricting along p_1 means requiring that the representations come from $G^\dagger$, corresponding to $U^\dagger \subset G^\dagger$, for which the homology of $\widehat{U}$ and $\widehat{U}^\dagger$ agree, i.e. the resulting sheaf is in the image of p_3^*. □

In particular, for the trivial infinitesimal character we get an equivalence

$$D(\mathrm{Fl}_{\mathrm{Betti}}) \simeq D(\mathrm{Fl}/(G^\dagger/B^\dagger)) \simeq D(*/G^\dagger)^{\chi=1}.$$

This gives an analytic version of the Beilinson–Bernstein localization on $\mathfrak{g}$-modules. However generally we have been interested in something with a little more structure: either locally analytic G-representations or $(\mathfrak{g}, K)$-modules (which we saw last time are essentially equivalent under reasonable finiteness conditions). We'll return to the latter in the next section, but for now we'd like to upgrade (analytic) $\mathfrak{g}$-modules to (locally analytic) G-representations. This corresponds to the sequence of maps from last time

$$(1 \subset G)^\wedge \to (1 \subset G)^\dagger \to G^{\mathrm{la}}$$

and the corresponding sequence of maps of classifying stacks: the first stack gives the algebraic version, the second the analytic theorem. To move to the third, observe that each point of the flag variety gives a unipotent subgroup $U^\dagger \subset G^\dagger \subset G^{\mathrm{la}}$ giving rise to a correspondence

$$*/G^{\mathrm{la}} \xleftarrow{a^{\mathrm{la}}} \mathrm{Fl}/G^{\mathrm{la}} \xrightarrow{b^{\mathrm{la}}} \mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger)$$

and a gerbe

$$\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger) \to \mathrm{Fl}/(G^{\mathrm{la}}/B^\dagger)$$

for $H^\dagger$ together with an isomorphism

$$\mathrm{Fl}/(G^{\mathrm{la}}/B^\dagger) \simeq \mathrm{Fl}^{\mathrm{an}}_{\mathrm{dR}}/(G^{\mathrm{la}}/G^\dagger) \simeq \mathrm{Fl}_{\mathrm{Betti}}/G_{\mathrm{Betti}}$$

by analytic Riemann–Hilbert, letting us work as above. Further we can extend the above map of correspondences by another:

$$\begin{array}{ccccc} */G^\dagger & \xleftarrow[a^\dagger]{} & \mathrm{Fl}/G^\dagger & \xrightarrow{b^\dagger} & \mathrm{Fl}/(G^\dagger/U^\dagger) \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} \\ */G^{\mathrm{la}} & \xleftarrow[a^{\mathrm{la}}]{} & \mathrm{Fl}/G^{\mathrm{la}} & \xrightarrow{b^{\mathrm{la}}} & \mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger) \end{array}$$

now with both squares Cartesian, and it follows from our work last time that the q_i^* are fully faithful. Therefore by base change along both squares we formally get the following group version. (Scholze inserts quotation marks around both stacks on the right; I guess this is because we haven't had an analogue to Lemma 1 in this setting, but it should be similar.)

Theorem 5. *The functor*

$$b_\sharp^\dagger a^{\dagger *} : D(*/G^{\mathrm{la}}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D(\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger))$$

is fully faithful, and its right adjoint $a_*^\dagger b^{\dagger *}$ is the Verdier quotient by sheaves whose pullbacks to Fl *have trivial cohomology. Restricting to regular weights, both functors become equivalences.*

In particular for the trivial infinitesimal character we get

$$D(\mathrm{Fl}_{\mathrm{Betti}}/G_{\mathrm{Betti}}) \simeq D(\mathrm{Fl}/(G^{\mathrm{la}}/B^\dagger)) \simeq D(*/G^{\mathrm{la}})^{\chi=1}.$$

Similar results hold for p-adic groups as well, using work of Rodrigues Jacinto–Rodríguez Camargo and Rodríguez Camargo on locally analytic representations and analytic de Rham stacks respectively, though lacking an analytic Riemann–Hilbert one has to work with (twisted) D-modules rather than Betti sheaves.

3. $(\mathfrak{g}, K)$-modules and the Matsuki correspondence

To get a version for $(\mathfrak{g}, K)$-modules, we fix algebraic groups G^{alg}, K^{alg} whose real points recover G and K respectively, and recall that $(\mathfrak{g}, K)$-modules can be described as $D(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge)$. The analogous setup to the above is the correspondence

$$*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \xleftarrow{a^{\mathrm{alg}}} \mathrm{Fl}/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \xrightarrow{b^{\mathrm{alg}}} \mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}),$$

recovering the algebraic setup of Theorem 3 for $K = \{1\}$. We have an $\widehat{H}$-gerbe

$$\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}) \to \mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{B})$$

and an isomorphism

$$\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{B}) \simeq \mathrm{Fl}_{\mathrm{dR}}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{G}) \simeq \mathrm{Fl}_{\mathrm{dR}}/K^{\mathrm{alg}}_{\mathrm{dR}}.$$

Applying similar ideas to above (relative to Theorem 3 gives the following:

Theorem 6. *The functor*

$$b_\sharp^{\mathrm{alg}} a^{\mathrm{alg}*} : D(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}))$$

is fully faithful, and its right adjoint $a_*^{\mathrm{alg}} b^{\mathrm{alg}*}$ is the Verdier quotient by sheaves whose pullbacks to Fl *have trivial cohomology. Restricting to regular weights, both functors become equivalences.*

In particular for the trivial infinitesimal character we get an equivalence

$$D(\mathrm{Fl}_{\mathrm{dR}}/K^{\mathrm{alg}}_{\mathrm{dR}}) \simeq D(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{B})) \simeq D(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge)^{\chi=1}.$$

Recalling that we have a relationship between $(\mathfrak{g}, K)$-modules and locally analytic G-representations, we would like to have a description of how this relationship interacts with

Beilinson–Bernstein localization: that is, what is the relationship between Theorems 5 and 6?

We have the following diagram of correspondences which essentially answers this question:

$$\begin{array}{ccccc}
*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge & \xleftarrow{a^{\mathrm{alg}}} & \mathrm{Fl}/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge & \xrightarrow{b^{\mathrm{alg}}} & \mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}) \\
{\scriptstyle a_{\mathrm{group}}}\uparrow & & \uparrow & & {\scriptstyle a_{\mathrm{loc}}}\uparrow \\
*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge & \longleftarrow & \mathrm{Fl}/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge & \longrightarrow & \mathrm{Fl}/((K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge/\widehat{U}) \\
\downarrow{\scriptstyle c_{\mathrm{group}}} & & \downarrow & & \downarrow{\scriptstyle c_{\mathrm{loc}}} \\
*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger & \longleftarrow & \mathrm{Fl}/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger & \longrightarrow & \mathrm{Fl}/((K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger/U^\dagger) \\
\downarrow{\scriptstyle d_{\mathrm{group}}} & & \downarrow & & \downarrow{\scriptstyle d_{\mathrm{loc}}} \\
*/G^{\mathrm{la}} & \xleftarrow{a^{\mathrm{la}}} & \mathrm{Fl}/G^{\mathrm{la}} & \xrightarrow{b^{\mathrm{la}}} & \mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger).
\end{array}$$

Using various geometric properties of these maps (primarily cohomological smoothness to ensure a left adjoint to pullback and properness of the structure sheaf to ensure it satisfies a (twisted) projection formula) together with base change and identifying the homology of $\widehat{U}$ and $U^\dagger$ as above gives the following result:

Proposition 7. *There is a natural equivalence*

$$d_{\mathrm{group}\sharp} c_{\mathrm{group}*} a^*_{\mathrm{group}} a^{\mathrm{alg}}_* b^{\mathrm{alg}*} \simeq a^{\mathrm{la}}_* b^{\mathrm{la}*} d_{\mathrm{loc}\sharp} c_{\mathrm{loc}*} a^*_{\mathrm{loc}}$$

of functors

$$D(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U})) \to D(*/G^{\mathrm{la}}).$$

This can be viewed as intertwining the functors of Theorems 5 and 6.

The classical Matsuki correspondence gives a bijection between the finite sets of K^{alg}-orbits and G-orbits on the flag variety Fl, together with an identification of the categories of sheaves on the quotients $\mathrm{Fl}/K^{\mathrm{alg}}$ and Fl/G. The full statement of the theorem (adapted for our setting) is not yet given in Scholze's draft notes, but the main novelty which can be viewed as an incarnation of the Matsuki correspondence for this point of view on $(\mathfrak{g}, K)$-modules is the following:

Theorem 8. *After restricting to the bounded part of $U(\mathfrak{h})$, the functor*

$$d_{\mathrm{loc}\sharp} c_{\mathrm{loc}*} a^*_{\mathrm{loc}} : D(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U})) \to D(\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger))$$

is an equivalence.

Proof sketch. By twisting by a line bundle we can restrict to the regular dominant locus in $U(\mathfrak{h})$, where by Beilinson–Bernstein localization and the compatibility of Proposition 7 this becomes a functor

$$D(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \to D(*/G^{\mathrm{la}}).$$

We have studied such a functor before: it was our functor $b'_! a^*$ from $(\mathfrak{g}, K)$-modules to locally analytic G-representations, and became an isomorphism after restriction to the bounded part of $Z(U(\mathfrak{g}))$. More properly, that result actually depends indirectly on this one, so to avoid circularity we should instead note that we could show directly that it was fully faithful, and then prove essential surjectivity by induction on strata. ∎

In particular, the proof that $b'_! a^*$ was an equivalence after localization to the bounded part of $Z(U(\mathfrak{g}))$—which we omitted at the time—goes roughly like this: one shows that $a_* b^*$ is its inverse on compact generators in $D(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge)$, and then claims that the image generates $D(*/G^{\mathrm{la}})$ (after suitable localization). The former part is explicit if tricky; the latter uses Beilinson–Bernstein localization. Indeed, $b'_! a^* = d_{\mathrm{group}\sharp} c_{\mathrm{group}*} a^*_{\mathrm{group}}$ in our language, and after suitable localization $a^{\mathrm{alg}}_* b^{\mathrm{alg}*}$ is an equivalence so it suffices to prove the claim for $d_{\mathrm{group}\sharp} c_{\mathrm{group}*} a^*_{\mathrm{group}} a^{\mathrm{alg}}_* b^{\mathrm{alg}*}$, which by Proposition 7 is $a^{\mathrm{la}}_* b^{\mathrm{la}*} d_{\mathrm{loc}\sharp} c_{\mathrm{loc}*} a^*_{\mathrm{loc}}$. Since $a^{\mathrm{la}}_* b^{\mathrm{la}*}$ is a Verdier quotient, it suffices to prove the result for $d_{\mathrm{loc}\sharp} c_{\mathrm{loc}*} a^*_{\mathrm{loc}}$, which follows from Theorem 8.

4. Discrete series

In the case where the infinitesimal character χ comes from that of a finite-dimensional representation, we get a G-equivariant line bundle $\mathcal{O}(\chi)$ on Fl and a (twisted) equivalence

$$D(\mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}) \simeq D(*/G^{\mathrm{la}})^\chi$$

sending a Betti sheaf $\mathcal{F}$ to $R\Gamma(\mathrm{Fl}, \mathcal{F} \otimes \mathcal{O}(\chi))$ with corresponding G-action.

Proposition 9. *For any complex $\mathcal{F}$ of constructible sheaves on $\mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}$, each cohomology group $H^i(\mathrm{Fl}, \mathcal{F} \otimes \mathcal{O}(\chi))$ is a quasiseparated dual nuclear Fréchet space with dense K-finite vectors and admissible $(\mathfrak{g}, K)$-module.*

Proof. By Theorems 4 and 8, this is equivalent to the claim for the corresponding locally analytic G-representation and its associated $(\mathfrak{g}, K)$-module, for which it follows from our results from last time. □

Let $U \subset \mathrm{Fl}$ be an open G-orbit. The proposition in the notes is incomplete, but I think it should read as follows:

Proposition 10. *We have $H^i_c(U, \mathcal{O}(\chi)) = 0$ unless $i = \dim \mathrm{Fl}$, in which case it is a discrete series representation of G (with infinitesimal character χ).*

More precisely, it is the minimal globalization of the discrete series representation; the maximal globalization is given by the complex conjugate of $R\Gamma(U, \omega_U(\chi))$. One can find explicit descriptions of the K-types and Harish-Chandra character on the regular semisimple elements. The proof is via applying the Matsuki correspondence to get to a situation where Beilinson–Bernstein localization is t-exact; I am not sure I understand this, but it seems plausible that one can directly compute on the algebraic side.