Analytic Riemann–Hilbert correspondence

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The usual Riemann–Hilbert correspondence on complex manifolds identifies regular holonomic D-modules with perverse sheaves. Our goal today is to give a stack-theoretic isomorphism reflecting and generalizing this correspondence in the world of analytic stacks, via transmutation. This will let us pass freely between D-modules and sheaves, which will be useful when discussing locally analytic representations and generally in the remainder of the course.

1. Betti stacks

We briefly recall the theory of Betti stacks from last time. We have a functor $\mathrm{Prof}^{\mathrm{light}} \to$ **AnStack** sending $S \mapsto \mathrm{AnSpec}\,\mathrm{Cont}(S, \mathbb{Z})$, which sends hypercovers to descendable !-hypercovers. Therefore it extends uniquely to light condensed sets (or anima), by descending the functor along a hypercover $T_\bullet \to S$ of any light condensed set (anima) S.

Composing with the condensification functor $S \mapsto \underline{S}$ from locally compact Hausdorff spaces to light condensed sets, we get a functor which we call the Betti stack S_{Betti}.

Proposition 1. *Let S be a metrizable compact Hausdorff space of finite (cohomological) dimension, and X be an analytic stack.*

(1) There is a natural equivalence

$$\mathcal{D}_{\mathrm{qc}}(S_{\mathrm{Betti}} \times X) \simeq D(S, \mathcal{D}(X)).$$

Here $\mathcal{D}(X) = \mathcal{D}_{\mathrm{qc}}(X)$ is the derived category of X, defined by descent from the categories of complete modules over analytic rings.

(2) Morphisms $X \to S_{\mathrm{Betti}}$ are equivalent to $D(\mathbb{Z})$-linear colimit-preserving symmetric monoidal functors

$$D(S, \mathbb{Z}) \to \mathcal{D}(X)$$

which, possibly after passing to a !-cover of X, preserve connective objects. These in turn are equivalent to collections of idempotent algebras $A_Z \in \mathcal{D}(X)$ for each closed subset $Z \subset S$ such that $Z \mapsto A_Z$ sends limits to colimits and finite unions to limits and (possibly after restriction to a !-cover) are connective.

These algebras A_Z should be thought of as the image under the functors $D(S, \mathbb{Z}) \to \mathcal{D}(X)$ of the idempotent objects $\mathbf{1}_Z \in D(S, \mathbb{Z})$.

We can understand one direction of the second part from the first part: since $\mathcal{D}_{\mathrm{qc}}(S_{\mathrm{Betti}}) \simeq D(S, \mathbb{Z})$, a morphism $X \to S_{\mathrm{Betti}}$ induces such a functor by pullback.

Proposition 2. *Let X be a complex-analytic space, viewed as an analytic stack. Then there is a natural surjection of analytic stacks $X \twoheadrightarrow X(\mathbb{C})_{\mathrm{Betti}}$.*

Proof. By Proposition 1, a map $X \to X(\mathbb{C})_{\mathrm{Betti}}$ is equivalent to a suitable collection of idempotent algebras $A_Z \in \mathcal{D}(X)$ for each closed subset $Z \subset X(\mathbb{C})$. For each such Z, we can consider the algebra of overconvergent functions $\mathcal{O}(Z)^\dagger$, and check that this satisfies the relevant conditions; so it remains to see that this map is surjective.

Choose a cover $S \to X(\mathbb{C})$ by a light profinite set $S = \lim_n S_n$. By the definition of $X(\mathbb{C})_{\mathrm{Betti}}$, it then admits a cover

$$\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}) \to X(\mathbb{C})_{\mathrm{Betti}},$$

so it suffices to prove surjectivity after base change to this cover, i.e. that

$$X \times_{X(\mathbb{C})_{\mathrm{Betti}}} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})$$

is surjective. For suitable choices of S, since X is locally affine we can take the left-hand side to be an affine analytic stack; indeed $S \to X(\mathbb{C})$ can be written as a sequential limit of $X_n \to X(\mathbb{C})$ for $X_n = \bigsqcup_{s \in S_n} \mathrm{Im}(S \times_{S_n} \{s\} \to X(\mathbb{C}))$, which we can assume are compact Stein spaces, and the left-hand side then becomes $\mathrm{AnSpec} \varinjlim_n \mathcal{O}(X_n)^\dagger$. Pullback induces a map $\mathcal{O}(X_n)^\dagger \to \mathrm{Cont}(S_n,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ splitting the map $\mathrm{Cont}(S_n,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathcal{O}(X_n)^\dagger$ coming from the map on analytic spectra as above, so the latter is descendable as is its sequential limit; in particular it follows that $X \times_{X(\mathbb{C})_{\mathrm{Betti}}} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})$ is a cover and so surjective, so the claim follows. □

2. Analytic de Rham stacks

The notion of de Rham stacks goes back to Simpson, who introduced for a smooth scheme X in characteristic 0 the de Rham stack X_{dR}, given by quotienting X by infinitesimal thickenings: the formal completion $\Delta(X)^\wedge \subset X \times X$ gives a relation on X, quotienting by which gives the de Rham stack $X_{\mathrm{dR}} = X/\Delta(X)^\wedge$. As a functor, this sends test objects S to $X(S_{\mathrm{red}})$; it satisfies the “transmutation” property identifying the derived category of D-modules on X with the derived category of quasicoherent sheaves on X_{dR}.

We can do the same construction when X is a complex manifold, viewed as an analytic stack over $\mathbb{C}_{\mathrm{gas}}$, via defining $\Delta(X)^\wedge$ to be the union of the infinitesimal thickenings of the diagonal and $X_{\mathrm{dR}} = X/\Delta(X)^\wedge$. For our purposes, though, it's better to take more analytic information into account: we should quotient by the overconvergent neighborhood of the diagonal, rather than just its formal neighborhood.

For any closed subset $Z \subset X$ in a complex manifold, we define the overconvergent neighborhood $(Z \subset X)^\dagger$ as $\lim_{U \supset Z} U$ where U ranges over open subsets of X containing Z and the limit is computed in the category of analytic stacks (i.e. implicitly applying the Betti stack functor). There is a canonical map $(Z \subset X)^\wedge \to (Z \subset X)^\dagger$. For the diagonal $\Delta : X \subset X \times X$, taking the overconvergent neighborhood $\Delta(X)^\dagger = (X \subset X \times X)^\dagger$ gives a relation in $X \times X$, quotienting by which gives a stack $X/\Delta(X)^\dagger$ which we call the analytic de Rham stack $X_{\mathrm{dR}}^{\mathrm{an}}$. The map $\Delta(X)^\wedge \to \Delta(X)^\dagger$ induces a map $g_X : X_{\mathrm{dR}} \to X_{\mathrm{dR}}^{\mathrm{an}}$.

A similar argument to the classical one relates D-modules to quasicoherent sheaves on X_{dR}; we would like to understand $\mathcal{D}_{\mathrm{qc}}(X_{\mathrm{dR}}^{\mathrm{an}})$ in terms of D-modules.

Proposition 3. *The structure sheaf $\mathcal{O}_{X_{dR}}$ is g_X-proper with g_X-proper dual $\mathcal{O}_{X_{dR}}[-2d_X]$, where d_X is the complex dimension of X. In particular, $g_{X*} \simeq g_{X!}[-2d_X]$ commutes with all colimits and satisfies the projection formula. Its left adjoint g_X^* is fully faithful.*

For our purposes, given a map $f : X \to Y$ and a sheaf $A \in D(X)$, A being f-proper with dual B essentially means that we have a natural isomorphism $f_*\underline{\mathrm{Hom}}(A, -) \simeq f_!(B \otimes -)$ of functors $D(X) \to D(Y)$. Specializing to the case of the proposition gives the identity $g_{X*} \simeq g_{X!}[-2d_X]$.

Proof sketch. When X is proper, the analogous statement holds replacing g_X by the structure map $X_{\mathrm{dR}} \to * = \mathrm{AnSpec}\,\mathbb{C}_{\mathrm{gas}}$, by Scholze's work on six functor formalisms on de Rham stacks. In this case $X_{\mathrm{dR}}^{\mathrm{an}} \to *$ is proper and so its $*$- and !-pushforwards agree, so the claim follows. In general, we work locally on a ball and compactify.

For the full faithfulness of g_X^*, it suffices to show that the unit $\mathrm{id} \to g_{X*}g_X^*$ is an equivalence. By the projection formula (which holds for g_{X*} by the first part) the right-hand side is $g_{X*}g_X^* - \simeq g_{X*}\mathcal{O}_{X_{\mathrm{dR}}} \otimes -$ so it suffices to show that $g_{X*}\mathcal{O}_{X_{\mathrm{dR}}} \simeq \mathcal{O}_{X_{\mathrm{dR}}^{\mathrm{an}}}$ or equivalently $g_{X!}\mathcal{O}_{X_{\mathrm{dR}}} \simeq \mathcal{O}_{X_{\mathrm{dR}}}^{\mathrm{an}}[2d_X]$. This is a local claim and reduces to the case of the affine line, which we will see more about shortly. □

Consider the case of the affine line $X = \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$, viewed as an analytic stack over $\mathbb{C}_{\mathrm{gas}}$; we will often write it as $\mathbb{G}_{\mathrm{a}}^{\mathrm{an}}$ to emphasize the group structure (under addition), leaving fact that we're over $\mathbb{C}$ implicit. Quotienting by the equivalence relation given by $\Delta(\mathbb{G}_{\mathrm{a}}^{\mathrm{an}})^\dagger$ is equivalent to quotienting by the subgroup stack $\mathbb{G}_{\mathrm{a}}^\dagger := (0 \subset \mathbb{G}_{\mathrm{a}}^{\mathrm{an}})^\dagger$, the overconvergent neighborhood of 0 (given by the analytic spectrum of the ring of gerbes of holomorphic functions at 0), so $\mathbb{G}_{\mathrm{a,dR}}^{\mathrm{an}} = \mathbb{G}_{\mathrm{a}}^{\mathrm{an}}/\mathbb{G}_{\mathrm{a}}^\dagger$. Similarly $\mathbb{G}_{\mathrm{a,dR}} = \mathbb{G}_{\mathrm{a}}^{\mathrm{an}}/\mathbb{G}_{\mathrm{a}}^\wedge$ where $\mathbb{G}_{\mathrm{a}}^\wedge = (0 \subset \mathbb{G}_{\mathrm{a}}^{\mathrm{an}})^\wedge$ is the formal neighborhood of the identity. In particular in this case the map $g_{\mathbb{G}_{\mathrm{a}}} : \mathbb{G}_{\mathrm{a,dR}} \to \mathbb{G}_{\mathrm{a,dR}}^{\mathrm{an}}$ is induced by $g : */\mathbb{G}_{\mathrm{a}}^\wedge \to */\mathbb{G}_{\mathrm{a}}^\dagger$.

The open immersion $j : \mathbb{G}_{\mathrm{a}}^{\mathrm{an}} \to \mathbb{G}_{\mathrm{a}}^{\mathrm{alg}}$, dual to the map from algebraic functions to analytic ones, induces a fully faithful functor

$$j_! : \mathcal{D}(\mathbb{G}_{\mathrm{a}}^{\mathrm{an}}) \to \mathcal{D}(\mathbb{C}_{\mathrm{gas}}[T]),$$

whose image consists of functions "vanishing at infinity," i.e. killed under tensor product with the idempotent algebra of functions at ∞, i.e. the subring of $\mathbb{C}((T^{-1}))$ of functions converging on a small punctured disc at ∞.

We have a similar story for the cover $g : */\mathbb{G}_{\mathrm{a}}^\wedge \to */\mathbb{G}_{\mathrm{a}}^\dagger$: we claim that pullback induces a fully faithful functor

$$\mathcal{D}_{\mathrm{qc}}(*/\mathbb{G}_{\mathrm{a}}^\dagger) \to \mathcal{D}_{\mathrm{qc}}(*/\mathbb{G}_{\mathrm{a}}^\wedge) \simeq D(\mathbb{C}_{\mathrm{gas}}[U])$$

with image that we can again describe as "vanishing at infinity," albeit in a slightly different sense. More precisely we have the following.

Proposition 4. *The pullback functor*

$$g^* : \mathcal{D}_{\mathrm{qc}}(*/\mathbb{G}_a^\dagger) \to \mathcal{D}_{\mathrm{qc}}(*/\mathbb{G}_a^\wedge) \simeq D(\mathbb{C}_{\mathrm{gas}}[U])$$

is fully faithful, and its image consists of modules killed under tensor product with the idempotent $\mathbb{C}_{\mathrm{gas}}[U]$*-algebra of power series*

$$\sum_{n \ll \infty} a_n U^n \in \mathbb{C}((U^{-1}))$$

for which there is some $r > 0$ *such that* $\lim_n |a_n| \frac{r^n}{n!} = 0$.

Proof. As in Proposition 3, we already know that $\mathcal{O}_{*/\mathbb{G}_{\mathrm{a}}^{\dagger}}$ is g-proper with dual $\mathcal{O}_{*/\mathbb{G}_{\mathrm{a}}^{\dagger}}[-2]$, and the full faithfulness of the pullback reduces to showing that $g_* \mathcal{O} \simeq \mathcal{O}$ or equivalently $g_! \mathcal{O} \simeq \mathcal{O}[2]$.

Write $h : * \to */\mathbb{G}_{\mathrm{a}}^{\wedge}$ for the quotient map. Then $h_! \mathcal{O}[1]$ is the regular representation of $\mathbb{G}_{\mathrm{a}}^{\wedge}$, which as a module over $\mathbb{C}[U]$ is $\mathbb{C}[T^{\pm 1}]/\mathbb{C}[T]$ with U acting by differentiation. In particular since U decreases the degrees of powers of T and we kill positive powers, the action of U is injective with cokernel given by constant functions, i.e. we have a short exact sequence

$$0 \to h_! \mathcal{O}[1] \xrightarrow{U} h_! \mathcal{O}[1] \to \mathcal{O} \to 0.$$

Applying $g_!$ gives an exact triangle

$$g_! h_! \mathcal{O}[1] \xrightarrow{U} g_! h_! \mathcal{O}[1] \to g_! \mathcal{O} \to 0.$$

Now, $g_! h_! = (g \circ h)_!$ is the pushforward along the quotient map $*/\mathbb{G}_{\mathrm{a}}^{\dagger}$ which is proper, and so $g_! h_! \mathcal{O} \simeq (g \circ h)_* \mathcal{O}$ is the regular representation of $\mathbb{G}_{\mathrm{a}}^{\dagger}$, on which the action of U by differentiation is surjective with kernel the constant representation $\mathcal{O}$; so the kernel of the corresponding map $g_! h_! \mathcal{O}[1] \to g_! h_! \mathcal{O}[1]$ is $\mathcal{O}[1]$, giving an exact triangle

$$g_! h_! \mathcal{O}[1] \to g_! h_! \mathcal{O}[1] \to \mathcal{O}[2].$$

Since the first map is the same as in the triangle above, this induces an isomorphism $g_! \mathcal{O} \simeq \mathcal{O}[2]$ as claimed, giving the full faithfulness of g^*. Note that this also completes the argument for the affine line mentioned above, by pullback.

The image of this functor must consist of modules killed under tensor product with some idempotent algebra for formal reasons. To compute this algebra, we observe that it should be the cone of $g^* g_* \mathbb{C}_{\mathrm{gas}}[U] \to \mathbb{C}_{\mathrm{gas}}[U]$. Since $g_* \mathbb{C}_{\mathrm{gas}}[U] \simeq g_! \mathbb{C}_{\mathrm{gas}}[U][-2]$, the regular representation of $\mathbb{G}_{\mathrm{a}}^{\dagger}$ shifted into degree 1, and so $g^* g_* \mathbb{C}_{\mathrm{gas}}[U]$ is this regular representation viewed as a $\mathbb{G}_{\mathrm{a}}^{\wedge}$-representation, i.e. the algebra of germs of holomorphic functions at $T = 0$, with U-action by differentiation. For the basis $\{T^n\}$, we can write $T^n = n! U^{-n} \cdot 1 = n! U^{-n}$, so enforcing U-overconvergence translates to the condition described. □

Combining these stories lets us study $\mathcal{D}_{\mathrm{qc}}(\mathbb{G}_{\mathrm{a,dR}}^{\mathrm{an}}) = \mathcal{D}_{\mathrm{qc}}(\mathbb{G}_{\mathrm{a}}^{\mathrm{an}}/\mathbb{G}_{\mathrm{a}}^{\dagger})$, taking the product of the pushforward and the pullback. This has image in a category of modules over a $\mathbb{C}_{\mathrm{gas}}$-algebra with generators T and U; these do not commute, but satisfy the relation $TU - UT = 1$, so we get a functor

$$\mathcal{D}_{\mathrm{qc}}(\mathbb{G}_{\mathrm{a,dR}}^{\mathrm{an}}) \to D(\mathbb{C}_{\mathrm{gas}}[T, U]/(TU - UT - 1)),$$

with image "vanishing at infinity" in both the T- and U-directions. The right-hand side is exactly modules over the Weyl algebra of algebraic differential operators, so we are giving an analytification of it in some sense.

3. The analytic Riemann–Hilbert correspondence

On the level of stacks, the analytic Riemann–Hilbert correspondence is the following statement:

Theorem 5. *Let X be a complex manifold. The map $X \to X_{\mathrm{Betti}}$ uniquely factors through X_{dR}^{an}, inducing an isomorphism $X_{dR}^{\mathrm{an}} \xrightarrow{\sim} X_{\mathrm{Betti}}$.*

Proof. First, since $Z_{\mathrm{Betti}} \simeq \varprojlim_{U \supset Z} U_{\mathrm{Betti}}$ we have

$$X \times_{X_{\mathrm{Betti}}} Z_{\mathrm{Betti}} \simeq X \times_{X_{\mathrm{Betti}}} \varprojlim_{U \supset Z} U_{\mathrm{Betti}} \simeq \varprojlim_{U \supset Z} X \times_{X_{\mathrm{Betti}}} U_{\mathrm{Betti}},$$

and since $X \to X_{\mathrm{Betti}}$ is a cover replacing U_{Betti} by $X \times_{X_{\mathrm{Betti}}} U_{\mathrm{Betti}}$ at most rearranges the limit over open sets containing Z and so the above is isomorphic to

$$\varprojlim_{U \supset Z} U_{\mathrm{Betti}} \simeq (Z \subset X)^{\dagger}.$$

In particular, it follows that

$$\Delta(X)^{\dagger} = (\Delta : X \subset X \times X)^{\dagger} \simeq (X \times X) \times_{(X \times X)_{\mathrm{Betti}}} X_{\mathrm{Betti}}.$$

Now, we have two surjections $X \to X_{\mathrm{Betti}}$ and $X \to X_{\mathrm{dR}}^{\mathrm{an}}$, which can be thought of as quotienting by the relations $X \times_{X_{\mathrm{Betti}}} X \subset X \times X$ and $\Delta(X)^{\dagger} \subset X \times X$, so it suffices to see that these relations agree. But by the above

$$X \times_{X_{\mathrm{Betti}}} X \simeq (X \times X) \times_{(X \times X)_{\mathrm{Betti}}} X_{\mathrm{Betti}} \simeq \Delta(X)^{\dagger}$$

and so the result follows. □

Finally, we'd like to recover something like the usual Riemann–Hilbert correspondence from this identification. We've mentioned that $\mathcal{D}_{\mathrm{qc}}(X_{\mathrm{dR}})$ is equivalent to the derived category of D-modules on X (in some formalisms by definition). One can then look at the subcategory of bounded complexes of regular holonomic D-modules, which we denote by $\mathcal{D}_{\mathrm{qc}}^{\mathrm{rh}}(X_{\mathrm{dR}})$. We have a functor $g_! : \mathcal{D}_{\mathrm{qc}}(X_{\mathrm{dR}}) \to \mathcal{D}_{\mathrm{qc}}(X_{\mathrm{dR}}^{\mathrm{an}})$, and Theorem 5 and Proposition 1 give isomorphisms $\mathcal{D}_{\mathrm{qc}}(X_{\mathrm{dR}}^{\mathrm{an}}) \simeq \mathcal{D}_{\mathrm{qc}}(X_{\mathrm{Betti}}) \simeq D(X, \mathbb{Z})$. Restricted to $\mathcal{D}_{\mathrm{qc}}^{\mathrm{rh}}(X_{\mathrm{dR}})$, the map $g_!$ is fully faithful, yielding a fully faithful functor

$$\mathcal{D}_{\mathrm{qc}}^{\mathrm{rh}}(X_{\mathrm{dR}}) \to D(X, \mathbb{Z}),$$

whose image can be described as the bounded complexes with Zariski-constructible cohomology. After shifts, $g_![-d_X] \simeq g_*[d_X]$ is t-exact for the standard t-structure on the left and the perverse t-structure on the right, so that in particular taking hearts gives an equivalence between regular holonomic D-modules and perverse sheaves on X.