Overview of analytic rings

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The goal today is to give a rapid introduction to (light) condensed mathematics, up through defining and seeing some examples of analytic rings, in particular the gaseous structure on $\mathbb{Z}[q]$. The material is drawn from Clausen–Scholze's lectures last year on analytic stacks as well as Juan's notes from his seminar, which is ongoing and which anyone interested in more details should attend. We will generally omit all proofs to focus on definitions, and will often be vague about details (e.g. we often elide the distinction between the derived and classical world, so probably the reader should imagine the word "animated" more often than it occurs below).

1. Light condensed sets and abelian groups

The first goal of the theory of condensed sets is to produce something approximating an abelian category of topological abelian groups. This fails with classical versions: the standard example is the map from $\mathbb{R}$ (equipped with the discrete topology) to $\mathbb{R}$ (with the standard topology), which is a continuous bijection and so in particular has trivial kernel and cokernel but not an isomorphism.

Condensed mathematics replaces the category of topological abelian groups with the category of condensed abelian groups: these are sheaves on the pro-étale site of a point. This definition, while pleasantly concise, is not especially useful at first glance, so let's elaborate a little: let Prof = Pro(Fin) be the category of profinite sets (equivalent to the category of totally disconnected compact Hausdorff spaces). This is equipped with a Grothendieck topology where the covers are finite jointly surjective families of maps, for which we can consider sheaves valued in a category $\mathcal{C}$; the category of such sheaves is $\mathrm{Cond}(\mathcal{C})$, condensed objects in $\mathcal{C}$ (e.g. condensed sets, condensed abelian groups, condensed rings...).

This definition is somewhat problematic in that the category of profinite sets is large, so various set-theoretic difficulties arise. This suggests the following modification, which will turn out to have better categorical properties as well in certain respects: we restrict to *metrizable* profinite sets, equivalently those which can be written as a countable limit of finite sets (equivalently the profinite sets admitting surjections from the Cantor set $\prod_{\mathbb{N}}\{0,1\}$). We call the resulting category $\mathrm{Prof}^{\mathrm{light}}$, with the corresponding topology, and the resulting sheaves $\mathrm{Cond}(\mathcal{C})^{\mathrm{light}}$ light condensed objects of $\mathcal{C}$. There is a condensification functor from topological spaces sending any topological space X to the light condensed set $\underline{X}$ sending

$$S \mapsto C(S, X)$$

for light profinite sets S. For X light profinite, this coincides with the Yoneda embedding. It has a left adjoint given by the "underlying topological space" sending a light condensed set T to the set $T(*)$, as a topological space given by

$$T(*)_{\mathrm{top}} = \operatorname*{colim}_{\underline{S}\to T} S.$$

We have explicit notions of quasicompact and quasiseparated objects in light condensed sets: T is quasicompact if it admits a surjection from a profinite set, and is quasiseparated if for every pair of maps $S \to T \leftarrow S'$ from profinite sets, the fiber product $S \times_T S'$ is quasicompact. The condensification functor restricted to metrizable compact Hausdorff spaces then gives an equivalence to qcqs light condensed sets.

We now turn to the category of light condensed abelian groups. By construction, this has some good formal properties: it is a Grothendieck abelian category, admits a natural tensor product (given by sheafification of the tensor product of presheaves on $\mathrm{Prof}^{\mathrm{light}}$) and internal Homs, and the forgetful functor to light condensed abelian sets has a left adjoint given by the free abelian group on a condensed set $T \mapsto (S \mapsto \mathbb{Z}[T(S)])$. Moreover it as claimed gives a natural replacement for the category of topological abelian groups in the following sense: the condensification functor extends to a commutative diagram

$$\begin{array}{ccc} \mathbf{TopAb} & \xrightarrow{\underline{(-)}} & \mathrm{Cond}(\mathbf{Ab})^{\mathrm{light}} \\ \downarrow & & \downarrow \\ \mathbf{Top} & \xrightarrow{\underline{(-)}} & \mathrm{Cond}(\mathbf{Set})^{\mathrm{light}} \end{array}$$

where the vertical maps are forgetful functors, i.e. if A is a topological abelian group then $\underline{A}$ has the structure of a light condensed abelian group. This already resolves our motivating problem: if $\mathbb{R}$ is the real numbers as an abelian group with the standard topology and $\mathbb{R}^\delta$ is the real numbers as an abelian group with the discrete topology, then we get an injection

$$\underline{\mathbb{R}^\delta} \to \underline{\mathbb{R}}$$

which is not an isomorphism. Its cokernel is the light condensed abelian group

$$\underline{\mathbb{R}}/\underline{\mathbb{R}^\delta} : S \mapsto C(S, \mathbb{R})/C^{lc}(S, \mathbb{R})$$

where $C^{lc}(-,-)$ is the space of locally constant functions.

The category of light condensed abelian groups has a further important property, which does not hold without restricting to the light setting: the object $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$, which we think of as parametrizing convergent sequences, is internally projective in light condensed abelian groups. We will make use of this fact later.

When we move on to studying analytic rings, we will want to be able to talk about various kinds of completeness structures on modules. The simplest kind is solidity, related to nonarchimedean completeness conditions.

The key idea of (light) solid abelian groups is that they should be the light condensed abelian groups in which summable sequences are the same as null sequences, similarly to the nonarchimedean case.

Let $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ as a light condensed abelian group, and for a condensed abelian group A we write $\mathrm{Null}(A) = \mathrm{Hom}(P, A)$, or $\underline{\mathrm{Null}}(A) = \underline{\mathrm{Hom}}(P, A)$ for the internal version. We think of this as the space of sequences in A converging to 0. If the map $\mathrm{Null}(A) \to (\mathbb{N}, A) = A(*)^{\mathbb{N}}$ is injective, then we can view this as a subspace of sequences in A, i.e. being a null sequence is a property of a sequence; in general however this map is not injective and being a null sequence is an additional structure.

The addition map on $\mathbb{N}$ makes P into an algebra $P \simeq \mathbb{Z}[\widehat{q}]$, with $\widehat{q}$ a variable which we think of as having $\lim_n \widehat{q}^n = 0$.

If $\{a_n\}$ is a null sequence, $\{a_n - a_{n+1}\}$ is summable in the sense that $\{\sum_{m>n}(a_m - a_{m+1})\}$ is a null sequence. More generally, a sequence is summable if the sequence of the sums of its tails is null; and we can always produce summable sequences from null sequences by telescoping. In the language above, this is given by an operator $1 - \mathrm{Shift}^*$ on $\underline{\mathrm{Null}}(A)$, where Shift is the endomorphism of P corresponding to shifting indices by 1. Thus the condition that all null sequences are summable can be rephrased as the condition that $1 - \mathrm{Shift}^* : \underline{\mathrm{Null}}(A) \to \underline{\mathrm{Null}}(A)$ is an isomorphism, in which case we say that A is solid. In particular if A is discrete then any null sequence has all but finitely many terms zero and so must be summable, i.e. discrete light condensed abelian groups are solid; in particular $\mathbb{Z}$ is solid.

The category of (light) solid abelian groups has a variety of good formal properties. In particular, it is a Grothendieck abelian category stable under limits, colimits, and extensions, and is the smallest such full subcategory of $\mathrm{Cond}(\mathbf{Ab})^{\mathrm{light}}$ containing $\mathbb{Z}$. Its inclusion into $\mathrm{Cond}(\mathbf{Ab})^{\mathrm{light}}$ admits a left adjoint $(-)^{\square}$ which we call solidification; there is a unique symmetric monoidal structure on it such that solidification is symmetric monoidal. Compatibly with our intuition that this is a nonarchimedean notion, $\mathbb{R}^{\square} = 0$. There are various other good properties, many on the derived level; another that's worth pointing out is that $\prod_{\mathbb{N}} \mathbb{Z}$ is flat in Solid, as well as a compact projective generator, which is not true without restricting to the light setting.

For a light profinite set $S = \lim S_n$, we write $\mathbb{Z}_{\square}[S] = \mathbb{Z}[S]^{\square}$, which is isomorphic to $\lim \mathbb{Z}[S_n]$; note that there is a unit map $S \to \mathbb{Z}_{\square}[S]$. We can equivalently say that a light condensed abelian group A is solid if for any light profinite set S, any map $S \to A$ factors through a unique map of light condensed abelian groups $\mathbb{Z}_{\square}[S] \to A$.

We finish this section by giving some examples of some tensor products of solid abelian groups. Two natural examples of complete abelian groups are power series rings and p-adic rings, both of which we can view as limits of $\mathbb{Z}[q]/q^n$ and $\mathbb{Z}/p^n$ respectively and so as solid. The solidification of $P \simeq \mathbb{Z}[\widehat{q}]$ is $\mathbb{Z}[[q]]$, and

$$\mathbb{Z}[[q_1]] \otimes_{\square}^{\mathbb{L}} \mathbb{Z}[[q_2]] \simeq \mathbb{Z}[[q_1, q_2]].$$

For each n we have a short exact sequence

$$0 \to \mathbb{Z}[q]/q^n \xrightarrow{q-p} \mathbb{Z}[q]/q^n \to \mathbb{Z}/p^n \to 0,$$

and taking the limit gives the short exact sequence

$$0 \to \mathbb{Z}[[q]] \xrightarrow{q-p} \mathbb{Z}[[q]] \to \mathbb{Z}_p \to 0$$

of solid abelian groups. This gives a resolution of $\mathbb{Z}_p \otimes_{\square}^{\mathbb{L}} \mathbb{Z}[[t]]$ by $\mathbb{Z}[[q]] \otimes_{\square}^{\mathbb{L}} \mathbb{Z}[[t]] \simeq \mathbb{Z}[[q,t]] \xrightarrow{q-p} \mathbb{Z}[[q]] \otimes_{\square}^{\mathbb{L}} \mathbb{Z}[[t]] \simeq \mathbb{Z}[[q,t]]$, i.e. it is just $\mathbb{Z}_p[[t]]$. Finally let ℓ be another prime and consider $\mathbb{Z}_p \otimes_{\square}^{\mathbb{L}} \mathbb{Z}_\ell$. This is represented by the complex

$$\mathbb{Z}_p[[q]] \xrightarrow{q-\ell} \mathbb{Z}_p[[q]],$$

which if $p \neq \ell$ is 0 since ℓ is a unit so so is $q - \ell$ and if $p = \ell$ is $\mathbb{Z}_p$ as above.

We mentioned above that $\mathbb{R}^{\square} = 0$, so solidity is the wrong notion for archimedean completeness. There is a good notion of liquid $\mathbb{R}$-vector spaces via defining for any $0 < p \leq 1$

a space of $< p$-convex measures $\mathcal{M}_{<p}(S)$ for each light profinite set S, and (similar to above for solid abelian groups) saying that a condensed $\mathbb{R}$-vector space V is p-liquid if every map $S \to V$ factors through a map of condensed $\mathbb{R}$-vector spaces $\mathcal{M}_{<p}(S) \to V$. We avoid the details as we will throughout the remainder of the course, using the formalism of *light* condensed sets to set up a gaseous version of the theory specializing to both the solid and liquid theories. (Notably this would be necessary for a global theory which would incorporate both archimedean and nonarchimedean places!)

2. Analytic rings

To do algebraic geometry-style analytic geometry, we want to have a category of something like topological rings with good categorical properties. Our experience thus far suggests passing to condensed rings. However, as with abelian groups, we also want to be able to think about completeness; and for a given condensed ring there may be many different ways to complete. One motivation is from the theory of adic spaces, which locally are given by Huber pairs, topological rings with subrings satisfying suitable properties which are to be thought of as rings of integral elements and for which we can study complete modules. Thus an analytic ring should be a condensed ring equipped with a suitable category of complete modules.

More precisely, a pre-analytic (or uncompleted analytic) ring is a pair $(A^{\triangleright}, \mathcal{D}(A))$ where $A^{\triangleright}$ is a condensed (animated) ring and $\mathcal{D}(A)$ is a full subcategory of the (derived infinity) category $\mathcal{D}(A^{\triangleright})$ of condensed $A^{\triangleright}$-modules, such that

- $\mathcal{D}(A)$ is stable under limits and colimits in $\mathcal{D}(A^{\triangleright})$, and the inclusion admits a left adjoint $A \otimes_{A^{\triangleright}} -$;
- for all (light) condensed abelian groups C and $M \in \mathcal{D}(A)$, the object $\underline{R\operatorname{Hom}}_{\mathbb{Z}}(C, M)$, viewed as an object of $\mathcal{D}(A^{\triangleright})$ via the $A^{\triangleright}$-action on M, is in $\mathcal{D}(A)$;
- $A \otimes_{A^{\triangleright}} -$ preserves connective objects, so that the t-structure on $\mathcal{D}(A^{\triangleright})$ induces one on $\mathcal{D}(A)$.

If $A^{\triangleright} \in \mathcal{D}(A)$ then we further call $(A^{\triangleright}, \mathcal{D}(A))$ an analytic ring. A morphism of (pre-)analytic rings $(A^{\triangleright}, \mathcal{D}(A)) \to (B^{\triangleright}, \mathcal{D}(B))$ is a map of animated condensed rings $A^{\triangleright} \to B^{\triangleright}$ such that the forgetful functor $\mathcal{D}(B^{\triangleright}) \to \mathcal{D}(A^{\triangleright})$ takes $\mathcal{D}(B)$ to $\mathcal{D}(A)$. There is a tensor product on $\mathcal{D}(A)$ given by $M \otimes_A N = (M \otimes_{A^{\triangleright}} N) \otimes_{A^{\triangleright}} A$.

Thus far, we've seen two analytic ring structures on the condensed ring $\mathbb{Z}$, i.e. suitable full subcategories of $\mathcal{D}(\mathrm{Cond}(\mathbf{Ab}))$, namely the whole category giving the initial analytic ring $\mathbb{Z} = (\mathbb{Z}, \mathcal{D}(\mathrm{Cond}(\mathbf{Ab})))$ and the solid subcategory giving the solid integers $\mathbb{Z}_{\square} = (\mathbb{Z}, \mathcal{D}(\mathrm{Solid}))$. More generally for any condensed ring R we can take the trivial analytic structure $(R, \mathcal{D}(R))$, and can construct solid analytic ring structures on any finite type $\mathbb{Z}$-algebra.

If $A = (A^{\triangleright}, \mathcal{D}(A))$ is a pre-analytic ring an B is an $A^{\triangleright}$-algebra, we get an induced pre-analytic ring structure on B, denoted $B_{A/}$, given by letting $\mathcal{D}(B_{A/}) \subset \mathcal{D}(B)$ be the subcategory of B-modules which under the forgetful functor to $A^{\triangleright}$-modules live in $\mathcal{D}(A)$. This makes the map $A^{\triangleright} \to B$ extend to a map of analytic rings $A \to B_{A/}$. It is the pushout

$(A^\triangleright, \mathcal{D}(A^\triangleright)) \otimes_{(A^\triangleright, \mathcal{D}(A))} (B, \mathcal{D}(B))$ of the trivial structures over A in the category of pre-analytic rings. If $B \in \mathcal{D}(A)$ as an A-module, then $B_{A/}$ is an analytic ring.

More generally, the category of pre-analytic rings has small colimits, with $B = \operatorname{colim} A_i$ having underlying condensed ring $B^\triangleright = \operatorname{colim} A_i^\triangleright$ and $\mathcal{D}(B) \subset \mathcal{D}(B^\triangleright)$ given by the B-modules whose restriction to each A_i is in $\mathcal{D}(A_i)$.

To recover analytic rings, we pass to a completion functor from pre-analytic rings to analytic rings. This is the left adjoint $A \mapsto A^=$. If $A = (A^\triangleright, \mathcal{D}(A))$, then $A^= = (A \otimes_{A^\triangleright} A^\triangleright, \mathcal{D}(A))$, i.e. the underlying ring is the A-completion of $A^\triangleright$ (i.e. the unit of $\mathcal{D}(A)$) and the category of modules is the same. In particular the category of analytic rings admits small colimits. Therefore to construct analytic rings it suffices to construct pre-analytic rings and then pass to the completion.

The other key example we want to discuss is the gaseous ring structure on $\mathbb{Z}[q]$. This is motivated from the Tate curve.

The Tate curve is an object $E = E_q$ over $\mathbb{Z}((q))$. We give two definitions, one analytic/geometric and one algebraic. The geometric definition is that E is given by taking $\mathbb{G}_{\mathrm{m}}$ over $\mathbb{Z}((q))$, which we think of as the space of T such that for some $n \in \mathbb{Z}$ we have $|q|^n \leq T \leq |q|^{-n}$, equipped with an action of q by multiplication, and quotient by this action: $E = \mathbb{G}_{\mathrm{m}/\mathbb{Z}((q))}/q^{\mathbb{Z}}$. This is one-dimensional and so for formal reasons will be algebraic over $\mathbb{Z}((q))$, giving a relative elliptic curve. We can write down explicit algebraic equations: it is given (on the affine locus) by the equation

$$y^2 + xy = x^3 - a_4 x - a_6$$

where

$$a_4 = \sum_{n \geq 0} 5n^3 \frac{q^n}{1 - q^n}, \qquad a_6 = \sum_{n \geq 0} \frac{7n^5 + 5n^3}{12} \frac{q^n}{1 - q^n}.$$

In particular (expanding the geometric series in q^n) we see that the coefficient of q^n in both a_4 and a_6 has polynomial growth in n. Thus we can specialize analytically to any q with $|q| < 1$ and get a convergent series, compatibly with the analytic description.

This suggests that more generally we should put some growth condition on our power series to ensure good analytic behavior, recovering polynomial growth in this case. Notably however polynomial growth is a tighter bound than is obviously necessary: to get convergence for $|q| < 1$, we just need the coefficients to have subexponential growth, and there is quite a lot of room in between (e.g. $\sum_{n \geq 0} e^{\sqrt{n}} q^{-n}$ still converges for $|q| < 1$). Where does the polynomial growth condition come from?

It turns out to fall out of the following formalism. Recall that we identified solid abelian groups A with those for which $1 - \mathrm{Shift}$ induces an isomorphism on $\underline{\mathrm{Hom}}(P, A)$. Identifying P with the algebra $\mathbb{Z}[\widehat{q}]$, the $1 - \mathrm{Shift}$ map is the same thing as multiplication by $1 - \widehat{q}$. Thus solid abelian groups are equivalently those for which $\underline{\mathrm{Hom}}(P/(1 - \widehat{q}), A) = 0$, which we can think of as localizing at $1 - \widehat{q}$.

More generally, $\mathbb{Z}_\square \otimes_\square \mathbb{Z}[\widehat{q}] \simeq \mathbb{Z}[[q]]$, and for a variable T we could ask for all null sequences to be T-summable, i.e.

$$\sum_n a_n T^n$$

to converge. We can get a T-summable sequence from any null sequence $\{a_n\}$ by passing to the T-twisted difference operator $\{a_n - Ta_{n+1}\}$, as then $\sum_{m>n}(a_m - Ta_{m+1})T^m = a_{n+1}T^{n+1} - a_{n+2}T^{m+2} + a_{n+2}T^{m+2} - \cdots = a_{n+1}T^{n+1}$ gives a null sequence for T powerbounded, so by the same logic as for solid abelian groups we should require $1 - qT$ invertible, i.e. localize at $1 - qT$, or equivalently require $\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[q]][T]/(1-qT), A) = 0$ for A to be $\mathbb{Z}[T]$-solid.

Repeating the same argument without passing to $\mathbb{Z}_\square$ suggests studying modules over the ring $P \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ after localization at $1 - \widehat{q}T$. These are the gaseous modules, giving the gaseous structure on $\mathbb{Z}[\widehat{q}]$ (after completion to get an analytic ring). Specializing to $T = 1$ recovers solid abelian groups. (This is a special case of a general idea of localizing categories; in particular under suitable assumptions the inclusion of modules on which a given endomorphism of a functor induces an automorphism has a left adjoint, in the above $T = 1$ case given by solidification.)

To form the Tate elliptic curve, we want q to be a topologically nilpotent unit, so we pass to $\mathbb{Z}((q)) = \mathbb{Z}[\widehat{q}][q^{-1}]$, with the gaseous structure the analytic ring structure induced from $\mathbb{Z}[\widehat{q}]$. It has underlying ring $\mathbb{Z}((q))^{\mathrm{gas},\triangleright}(*)$ given by Laurent series $\sum_n a_n q^n$ with $a_n = 0$ for n sufficiently negative and $|a_n|$ having at most polynomial growth. (Scholze sketches (in lecture 14 of his video lectures with Clausen) where the polynomial growth condition comes from, but I did not follow the explanation very well; it seems like from more straightforward boundedness conditions on each test finite set, we get precise bounds on the coefficients (in terms of binomial coefficients), which letting the sets vary then can be up to arbitrary polynomials but not higher (more precisely there must exist $m, k > 0$ such that $\lim_{n\to\infty} |a_n|(n+m)^{-k} = 0$).)

We also get a gaseous structure on the real numbers. Let V be a condensed $\mathbb{R}$-vector space; we say that it is gaseous if $1 - \frac{1}{2}\mathrm{Shift}$ induces an isomorphism on $\underline{\mathrm{Hom}}_{\mathbb{Z}}(P, V)$. (This choice of $\frac{1}{2}$ is analogous to the choice of parameter in the liquid setting, but here it doesn't matter: any parameter in $(0,1)$ gives the same theory, as $1 - q^n \cdot \mathrm{Shift}$ induces an isomorphism iff $1 - q \cdot \mathrm{Shift}$ does.) In particular any p-liquid vector space is gaseous for any p; surprisingly this larger category still gives a good theory.

For example, we can understand $\mathbb{R}^{\mathrm{gas}}$ to be (the completion of) $(\mathbb{Z}[\widehat{q}][q^{-1}], \mathbf{Mod}^{\mathrm{gas}})/(1 - 2q)$, i.e. forcing $q = \frac{1}{2}$. One can write down an explicit description of the resulting $\mathbb{R}^{\mathrm{gas}}[S]$, formally similar to the description for the liquid case.

Finally, we mention the Berkovich picture. In the real case, we have a family of α-liquid structures for $\alpha \in (0,1]$, and we can think of the gaseous case as the limit as $\alpha \to 0$. We can make all the same definitions (here omitted) in the nonarchimedean case, and in fact the ultrametric inequality means that we can define α-liquid structures for any $\alpha \in (0, \infty)$. We can again study the gaseous limit as $p \to 0$, and in this case can also look at the case $\alpha \to \infty$; this recovers the solid situation. So the gaseous structure is "global" in a way that none of the structures we've seen previously are, as a limit point of structures arising from both archimedean and nonarchimedean places. This picture is closely analogous to the Berkovich space of $\mathbb{Z}$, parametrizing families of absolute values on $\mathbb{Z}$: for each prime, a p-adic absolute value $|-|_p^\alpha$ for $\alpha \in (0, \infty)$, corresponding to the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}_p$, and in the "limit at infinity" the trivial absolute value on $\mathbb{F}_p$ pulled back along the map from $\mathbb{Z}$; the archimedean absolute values $|-|_\infty^\alpha$ for $\alpha \in (0,1]$), corresponding to $\mathbb{Z} \hookrightarrow \mathbb{R}$; and the trivial absolute value on $\mathbb{Z}$, corresponding to $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Similarly we should be able to recover all of the solid and liquid structures on each local field from the gaseous structure on $\mathbb{Z}[\widehat{q}]$.