Overview of analytic stacks

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1. !-descent

The goal of this talk is to define (and say something about) a geometric category of analytic stacks, in which we can do a version of analytic geometry subsuming and extending algebraic geometry but with application to "analytic" situations, e.g. formal schemes, adic spaces, or archimedean/nonarchimedean analytic geometry.

Just as in classical algebraic geometry, our stacks should be some sort of sheaves on the opposite category of analytic rings (which we discussed last time). The first and most important question is: which Grothendieck topology should we put on $\mathrm{AnRing}^{\mathrm{op}}$? This should be some topology allowing us to glue together the affine objects $\mathrm{AnSpec}\, A$ to global ones in a way compatible with the various special cases that analytic stacks are supposed to subsume, going back to topological spaces as (light) condensed sets.

One very natural desideratum is that we should have the following structure sheaf: if $A = (A^{\triangleright}, \mathcal{D}(A))$ is an analytic ring, we have a natural assignment $A \mapsto \mathcal{D}(A)$. We would like this to form a sheaf of ∞-categories for our topology, i.e. we should get an ∞-category $\mathcal{D}(X)$ for any analytic stack X by descent. Further $X \mapsto \mathcal{D}(X)$ should underlie a six functor formalism, which most simply are pairs of adjoint functors $\otimes$, $\underline{R\,\mathrm{Hom}}$, $-^*$, $-_*$, and (in certain cases) $-_!$, $-^!$ satisfying various compatibilities, e.g. base change and a projection formula.

To give a little more detail about what "certain cases" means, recall how !-functors are usually defined, say on (derived) étale sheaves. We first assume we have a "four-functor formalism," i.e. a symmetric monoidal structure and internal Homs together with a base change functor $-^*$ and its right adjoint $-_*$. Consider a map $f : X \to Y$. There are two relevant special cases: if f is proper, then the right adjoint f_* has a further right adjoint $f^!$; and if f is an open immersion, f^* has a further left adjoint $f_!$. (In fact, this is true for all f smooth, but in general it will be a different functor from the one we're looking for.) In the first case, we let $f_! = f_*$, the left adjoint of $f^!$; in the second we let $f^! = f^*$, the right adjoint of $f_!$, so that in both cases we have an adjoint pair $(f_!, f^!)$.

More generally, we can try to define $f_!$ for any map f which can be written as a composite of open immersions and proper maps. A particularly natural way is via (relative) compactifications: if f factors as

$$X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y$$

with j an open immersion and $\overline{f}$ proper, then we set $f_! = \overline{f}_* j_!$. A priori this depends on the choice of compactification, but one can generally show that the result is independent of it.

However, one should not expect all maps to be able to be written in this form—for one thing open immersions and proper maps are both separable, so f must be separable as well; it should also have finite type. More abstractly, one could say following Scholze that it should be "compactifiable"; more abstractly yet, one can notice that this really just requires isolating two classes of maps I and P (in our case above open immersions and proper maps) such that for $f \in I$ the pullback f^* has a left adjoint $f_!$ and for $f \in P$ the pushforward f_* has a right adjoint $f^!$, and then using these to define $(f_!, f^!)$ for maps which can be written as a composite of these. Using the formalism of Lucas Mann (refining Liu–Zheng), one can verify that this gives a well-defined pair of adjoint functors $(f_!, f^!)$ fitting into a six functor formalism (so e.g. satisfying base change and a projection formula).

So in our case, we would like to have some notion of what open immersions and proper maps should be. These will both be notably different from the usual geometric notions; instead we will take the cohomological desiderata and try to directly define classes of maps (between affine analytic stacks) with these properties, which will not always align with the classical versions.

One property that proper maps should have is that they should satisfy a projection formula on $*$-pushforward: $f_*(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_*\mathcal{F} \otimes \mathcal{G}$. Given a full six functor formalism, this is because this should be true in general for !-pushforward, and for f proper we have $f_! = f_*$. However we can also take it as a definition of sorts: for f a map of (affine) analytic stacks, we say that it is proper if it satisfies $f_*(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_*\mathcal{F} \otimes \mathcal{G}$ naturally in $\mathcal{F}$ and $\mathcal{G}$.

This is, by design, significantly weaker than the usual notion of properness, as we'll see shortly: maps of affine schemes are proper only if they're finite, and we want more general maps for our cohomological properness.

For any $f : \operatorname{AnSpec} B \to \operatorname{AnSpec} A$ corresponding to a map of analytic rings $A \to B$, the map of underlying condensed rings $A^\triangleright \to B^\triangleright$ together with the analytic ring structure on A gives a map to the induced analytic ring structure $A \to B_{A/}$ factoring $A \to B$. We claim that f is proper if and only if the induced $B_{A/} \to B$ is an isomorphism, i.e. if B has the induced analytic ring structure.

Before we look at the proof, we observe that this already gives a much larger class of proper maps than we would have naively: for discrete rings, if we equipped them with the trivial analytic ring structure than for any map of discrete rings $A \to B$ the trivial analytic structure on B is induced from that on A and so every map of affine schemes, viewed in this way as affine analytic stacks, is proper! However this notion of properness does agree with that of Huber for adic spaces.

To prove the claim, let $M \in \mathcal{D}(A)$, $N \in \mathcal{D}(B)$; we want to know whether $f_*(f^*M \otimes_B N)$ is the same thing as $M \otimes_A f_*N$, i.e. whether $(M \otimes_A B) \otimes_B N$, viewed as an A-module by restriction, is the same thing as $M \otimes_A N|_A$. If B has the induced analytic ring structure, then we can view the tensor product as the one in $\mathcal{D}(A)$, and so this is formal. Conversely, assume it holds, and take $N = B^\triangleright$, so tensoring with it over B is the identity; then this is the statement that $M \otimes_A B \simeq M \otimes_A B^\triangleright$ for any $M \in \mathcal{D}(A)$, and therefore B has the induced analytic ring structure: taking $M = A^\triangleright$ gives $A^\triangleright \otimes_A B \simeq A^\triangleright \otimes_A B^\triangleright$ which in the category of analytic rings gives $B \simeq B_{A/}$. In particular, the class of proper maps is stable under base change, and there is a proper base change formula.

Consider for example $\operatorname{AnSpec}(\mathbb{Z}[T], \mathbb{Z})_\square \to \operatorname{AnSpec} \mathbb{Z}_\square$. A priori, this is the $\mathbb{Z}$-solid affine line and so we would not expect it to be proper in the classical sense; but in our sense it will be, since $(\mathbb{Z}[T], \mathbb{Z})_\square$ has the analytic ring structure induced from $\mathbb{Z}_\square$.

Similarly, we'll define open immersions to be maps $j : \operatorname{AnSpec} B \to \operatorname{AnSpec} A$ such that j^* admits a left adjoint $j_!$ satisfying a projection formula $j_!(\mathcal{F} \otimes j^*\mathcal{G}) \simeq j_!\mathcal{F} \otimes \mathcal{G}$ naturally in $\mathcal{F}$ and $\mathcal{G}$. Notably open immersions in algebraic geometry are *not* open immersions in this sense, unless they are also closed immersions (they have the right cohomological properties on e.g. étale sheaves but not quasi-coherent sheaves). Again however in adic geometry the definitions are compatible: e.g. $\mathbb{G}_m \to \mathbb{A}^1$ in the algebraic setting is not an open immersion in our sense, but $\operatorname{AnSpec} \mathbb{Z}[T^{\pm 1}]_\square \to \operatorname{AnSpec} \mathbb{Z}[T]_\square$ is an example. A slightly more subtle example is $j : \mathrm{AnSpec}\,\mathbb{Z}[T]_\square \to \mathrm{AnSpec}(\mathbb{Z}[T],\mathbb{Z})_\square$ is an open immersion. Indeed, for $M \in \mathcal{D}((\mathbb{Z}[T],\mathbb{Z})_\square)$, j_*j^*M is its $\mathbb{Z}[T]_\square$-solidification, which is $\underline{R\,\mathrm{Hom}}_{\mathbb{Z}[T]}([\mathbb{Z}[T] \to \mathbb{Z}((T))], M)$. The left adjoint should satisfy $\mathrm{Hom}(j_!j^*M, N) = \mathrm{Hom}(j^*M, j^*N) = \mathrm{Hom}(M, j_*j^*N)$ and so by tensor-hom adjunction should be $M \otimes^{\mathbb{L}}_{(\mathbb{Z}[T],\mathbb{Z})_\square} [\mathbb{Z}[T] \to \mathbb{Z}((T))]$, compatibly with the projection formula $j_!j^*M \simeq j_!\mathcal{O} \otimes M$, with $j_!\mathcal{O} = [\mathbb{Z}[T] \to \mathbb{Z}((T))]$. We think of this as "compactly supported" regular functions in the sense of vanishing near ∞. In general, we again get a class of open immersions stable under base change.

We can plug these classes into the Mann–Liu–Zheng machinery to produce a class of !-able maps which can be written as composites of these proper maps and open immersions, checking some basic compatibilities. By construction these satisfy a projection formula and base change (extending proper base change). For example, $\mathrm{AnSpec}\,\mathbb{Z}[T]_\square \to \mathrm{AnSpec}\,\mathbb{Z}_\square$ is neither an open immersion nor proper; however via our examples above we can factor it as

$$\mathrm{AnSpec}\,\mathbb{Z}[T]_\square \to \mathrm{AnSpec}(\mathbb{Z}[T],\mathbb{Z})_\square \to \mathrm{AnSpec}\,\mathbb{Z}_\square$$

with the first map an open immersion and the second map proper; this is a version in this setting of compactification.

Returning to the question of the choice of Grothendieck topology, the idea is that the covers should be maps that satisfy $*$-universal descent and !-universal descent. What does this mean? For $? \in \{*, !\}$, a map $f : Y \to X$ satisfies ?-descent if the resulting map

$$\mathcal{D}(X) \xrightarrow{f^?} \lim_{\Delta} \left(\mathcal{D}(Y) \underset{p_2^?}{\overset{p_1^?}{\rightrightarrows}} \mathcal{D}(Y \times_X Y) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \cdots \right)$$

is an isomorphism, and satisfies ?-universal descent if the same holds after any base change.

This seems like a lot to ask for; fortunately we have to check much less. It turns out that if f satisfies !-descent, then it satisfies universal $*$- and !-descent, and so it suffices to take covers to be !-able maps satisfying !-descent (and finite disjoint unions thereof). This includes for example all countably presentable faithfully flat maps and quotients by nilpotent ideals; more surprisingly it includes all h-covers (i.e. universally submersive maps, including fppf covers and proper maps of finite presentation which are isomorphisms away from a closed substack), and restricted to Noetherian schemes this is equivalent to the h-topology.

The goal of having this powerful topology is to compare different constructions which we expect to recover the same analytic stack; we'll see some examples later. This is essentially the strongest topology on which we'll have this kind of descent.

2. Analytic stacks

An analytic stack is then a(n accessible) functor $X : \mathrm{AnRing} \to \mathbf{Anima}$ commuting with finite products such that for any hypercover $\mathrm{AnSpec}(A_\bullet) \to \mathrm{AnSpec}(A)$ for which

$$\mathcal{D}(A) \simeq \lim{}^! \mathcal{D}(A_\bullet)$$

we have

$$X(A) \simeq \lim X(A_\bullet),$$

where $\lim^!$ denotes the limit of the simplicial diagram as above induced by !-pullbacks. This should be thought of as establishing an ∞-topos between !-sheaves (where the condition is automatic) and !-hypersheaves (where we would not require it); we'll come back to why this is a good choice later. A simple example is analytic rings by the Yoneda embedding, with $\mathrm{AnSpec}(A)$ sending $B \mapsto \mathrm{Hom}(A, B)$. Small colimits of affine analytic stacks likewise give (accessible) analytic stacks.

Restricting to discrete rings (via equipping them with the trivial analytic structure) generates a fully faithful functor from schemes, and in fact derived schemes, to analytic stacks. (The full faithfulness of this functor is nontrivial to prove, and uses Bhatt's Tannaka duality, an identification

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Fun}_{\otimes}^{\mathrm{ex}}(D_{\mathrm{perf}}(Y), D_{\mathrm{perf}}(X)) \simeq \mathrm{Fun}_{\otimes}^{\mathrm{cocont}}(D(Y), D(X))$$

for X and Y algebraic spaces with Y qcqs.)

Another example is given by adic spaces: for a Huber pair (A, A^+), one can define an analytic ring $(A, A^+)_\square$, for which roughly the complete modules are the A-modules which are $A^+_\square$-complete. Thus each sheafy affine adic space $\mathrm{Spa}(A, A^+)$ gives an affine analytic stack $\mathrm{AnSpec}(A, A^+)_\square$, and by descent we get a functor from adic spaces to analytic stacks.

Note that there are two functors from schemes to adic spaces, given on affine schemes by $\mathrm{Spec}\, R \mapsto \mathrm{Spa}(R, R)$ and $\mathrm{Spec}\, R \mapsto \mathrm{Spa}(R, \mathbb{Z})$, which give different results under passing to analytic stacks.

This actually gives us three functors from schemes to analytic stacks: one with trivial analytic structure $\mathrm{Spec}\, R \mapsto \mathrm{AnSpec}\, R$, one with the analytic ring structure induced from the solid structure on $\mathbb{Z}$, $\mathrm{Spec}\, R \mapsto \mathrm{AnSpec}(R, \mathbb{Z})_\square$, and one with the solid structure $\mathrm{Spec}\, R \mapsto \mathrm{AnSpec}(R, R)_\square$. All give fully faithful embeddings, so we need to distinguish between these various incarnations of schemes in the analytic world. The relative solid structure over $\mathbb{Z}$ can be viewed as base change $- \times_{\mathrm{AnSpec}\, \mathbb{Z}} \mathrm{AnSpec}\, \mathbb{Z}_\square$. More generally, this construction gives for any analytic ring A a base change functor from derived schemes over $A^{\triangleright}(*)$ to analytic stacks over A.

Another especially relevant case for us is complex analytic spaces, which can be incarnated as analytic stacks over $\mathbb{C}_{\mathrm{gas}}$, the gaseous structure on $\mathbb{C}$. We think of the affine pieces as compact Stein subspaces and the structure sheaf as the sheaf of overconvergent functions, i.e. with values on a disk D given by the colimit of holomorphic functions on strictly larger disks. This gives a "dual nuclear Frechet" space (and $\mathbb{C}$-algebra), which is formally a gaseous $\mathbb{C}$-algebra; this then gives an analytic stack locally over $\mathbb{C}_{\mathrm{gas}}$, and globally by gluing. A similar story works for real manifolds, including on smooth/k-differentiable/continuous functions with more care, recovering real or complex manifolds of the appropriate type.

A different sort of specialization is to condensed sets, or let's say (light) condensed anima. In fact this already exists on usual stacks: send a profinite set S to $\mathrm{Spec}\, \mathrm{Cont}(S, \mathbb{Z})$, which for S finite is just finitely many disjoint copies of $\mathrm{Spec}\, \mathbb{Z}$ and so in general is a limit of such things. Then we can map back into analytic stacks: $S \mapsto \mathrm{AnSpec}\, \mathrm{Cont}(S, \mathbb{Z})$.

Now we can justify our choice of this intermediate ∞-topos between sheaves and hypersheaves: (light) condensed anima are by definition hypersheaves of anima on light profinite sets for the standard Grothendieck topology, so for this functor any hypercover of light profinite sets should be mapped to something for which we enforce descent. It definitely is mapped to a hypercover; the extra condition that we impose is just that descent in fact holds. (The lightness condition here is necessary to ensure that a surjection $S' \twoheadrightarrow S$ induces a *countably* presentable faithfully flat map $\mathrm{Cont}(S,\mathbb{Z}) \to \mathrm{Cont}(S',\mathbb{Z})$.)

Just as there are multiple embeddings of schemes into analytic stacks, we likewise have multiple ways of embedding (say topological) manifolds. One is, as above, to look at the algebra of continuous functions, treat it as an analytic ring, and take its spectrum (and glue); the other would be to just view the manifold as a topological space and take its condensification, and then embed the resulting condensed set. This gives something different, which is actually defined over $\mathbb{Z}$. There is however a map between the two. Consider the example of the sphere S^2 as a topological manifold. This gives an (gaseous) analytic space, in this case just $\mathrm{AnSpec}\,\mathrm{Cont}(S^2,\mathbb{C})_{\mathrm{gas}}$ (working with complex coefficients here).

On the other hand, we could take S^2 viewed as a real-analytic manifold, still over $\mathbb{C}_{\mathrm{gas}}$, given by $\mathrm{AnSpec}\, C^\omega(S^2,\mathbb{C})_{\mathrm{gas}}$, where C^ω denotes real-analytic functions.

Going further, we could view S^2 as a complex-analytic space, where in fact it is isomorphic to the projective line $\mathbb{P}^1_{\mathbb{C}_{\mathrm{gas}}}$. This is no longer affine: it is glued from two copies of $\mathrm{AnSpec}\,\mathcal{O}(\mathbb{D})^\dagger$, the spectrum of overconvergent functions on a disk.

Even further, we could take the algebraic $\mathbb{P}^1_{\mathbb{C}_{\mathrm{gas}}}$, i.e. the image of the schematic $\mathbb{P}^1$ over $\mathbb{C}_{\mathrm{gas}}$, viewed as glued from two copies of $\mathrm{AnSpec}\,\mathbb{C}[T]_{\mathrm{gas}}$.

Finally, we could take S^2 as a topological space and thence as a condensed set, and base change to $\mathbb{C}_{\mathrm{gas}}$. Here we take locally constant functions, which don't exist naively but do after profinite covers. The inclusions of locally constant functions into regular functions into overconvergent functions into (restrictions of) real-analytic functions into continuous functions gives a series of maps

$$(S^2)^{\text{top man}}_{\mathbb{C}_{\mathrm{gas}}} \to (S^2)^{\mathbb{R}\text{-an}}_{\mathbb{C}_{\mathrm{gas}}} \to (S^2)^{\mathbb{C}\text{-an}}_{\mathbb{C}_{\mathrm{gas}}} = (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}} \to (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{sch}}_{\mathbb{C}_{\mathrm{gas}}} \to (S^2)^{\mathrm{cond}}_{\mathbb{C}_{\mathrm{gas}}}.$$

A version of GAGA says that the third map is an isomorphism of analytic stacks! We will often use this sort of GAGA statement to pass between algebraic and analytic incarnations of complex-analytic objects.

More generally, if X is a locally compact Hausdorff space with finite dimension and A is an analytic ring, there is an isomorphism

$$\mathcal{D}(X^{\mathrm{cond}} \times \mathrm{AnSpec}\,A) \simeq \mathrm{Sh}(X,\mathcal{D}(A)).$$

This is one of the key properties of the Betti stack; let's say something more about it before we try to prove the above equivalence.

Given X as above, say compact, we can find a profinite set T_0 surjecting onto X. This gives a simplicial diagram

$$X \leftarrow T_0 \leftleftarrows T_1 \Lleftarrow \cdots$$

inducing a similar diagram on $\mathrm{Spec}\,\mathrm{Cont}(T_i,\mathbb{Z})$; we write X^{Betti} for the colimit

$$X^{\mathrm{Betti}} \leftarrow \mathrm{Spec}\,\mathrm{Cont}(T_0,\mathbb{Z}) \leftleftarrows \mathrm{Spec}\,\mathrm{Cont}(T_1,\mathbb{Z}) \Lleftarrow \cdots .$$

The key fact about the Betti stack is a special case of the above: the quasi-coherent sheaves on $X^{\mathrm{Betti}} = X^{\mathrm{cond}}$ are sheaves of abelian groups on X.

We can also ask what the functor of points description of X^{Betti} should be. We claim that for an analytic ring R, maps $\operatorname{AnSpec} R \to X^{\mathrm{Betti}}$ should be equivalent to symmetric monoidal colimit-preserving $\mathcal{D}(\mathbb{Z})$-linear functors $F : \mathrm{Sh}(X, \mathcal{D}(\mathbb{Z})) \to \mathcal{D}(R)$ such that !-locally the image of any connective object is connective.

The first claim above, $\mathcal{D}(X^{\mathrm{Betti}} \times \operatorname{AnSpec} R) \simeq \mathrm{Sh}(X, \mathcal{D}(R))$, is now relatively straightforward given the strength of !-descent. Consider first the case where X is a point. Then $X^{\mathrm{Betti}} \simeq \operatorname{Spec} \mathbb{Z}$ and so this is just the identity $\mathcal{D}(\operatorname{AnSpec} R) = \mathrm{Sh}(*, \mathcal{D}(R)) = \mathcal{D}(R)$, which is true essentially by definition. Since everything is compatible with finite disjoint unions, we get the result for X finite as well; and by descent along light profinite covers we get it for light profinite sets as well.

Finally, for X arbitrary with a simplicial profinite cover $T_\bullet \to X$ as above, we get a !-cover $\operatorname{Spec} \operatorname{Cont}(T_\bullet, \mathbb{Z}) = T_\bullet^{\mathrm{Betti}}$, which we know satisfies $\mathcal{D}(T_\bullet^{\mathrm{Betti}} \times \operatorname{AnSpec} R) \simeq \mathrm{Sh}(T_\bullet, \mathcal{D}(\mathbb{R}))$. Thus we can descend both sides to get the desired statement.

Given this description, one direction of the statement about the functor of points is now easy: given a map $\operatorname{AnSpec} R \to X^{\mathrm{Betti}}$, we get a pullback functor $\mathcal{D}(X^{\mathrm{Betti}}) \simeq \mathrm{Sh}(X, \mathcal{D}(\mathbb{Z})) \to \mathcal{D}(R)$, which one can check has the desired properties. To go the other direction, take a hypercover $T_\bullet \to X$; this induces a simplicial complex of the pushforwards along $\pi_i : T_i \to X$ of the constant sheaves

$$\mathbb{Z} \to \ \pi_{0*}\mathbb{Z} \rightrightarrows \pi_{1*}\mathbb{Z} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots$$

on X, which is the co-Čech nerve of $\mathbb{Z} \to \pi_{0*}\mathbb{Z}$. Applying F gives an $F(\mathbb{Z}) = R^{\triangleright}$-algebra $F(\pi_{0*}\mathbb{Z})$, and equipping it with the induced analytic structure from R gives an analytic R-algebra R', whose co-Čech nerve maps (after taking AnSpec) to that of $\operatorname{Spec} \operatorname{Cont}(T_0, \mathbb{Z}) \to X^{\mathrm{Betti}}$. One can show that $R \to R'$ is a !-cover (in fact proper, though this takes some nontrivial work) and so this induces a map $\operatorname{AnSpec} R \to X^{\mathrm{Betti}}$ as desired.

A particularly important example of an analytic stack that we'll see in the rest of the course is the classifying stack of a (real) Lie group: for a Lie group G, i.e. a group object in real-analytic manifolds, we let G^{la} denote the corresponding group object in analytic stacks via realizing G as a real-analytic manifold as for S^2 above. Then we can look at $*/G^{\mathrm{la}} = \operatorname{AnSpec}(\mathbb{R}_{\mathrm{gas}})/G^{\mathrm{la}}$. The quotient map $f : * \to */G^{\mathrm{la}}$ induces two maps $f^*, f^! : \mathcal{D}(*/G^{\mathrm{la}}) \to \mathcal{D}(*) = \mathcal{D}(\mathbb{R}_{\mathrm{gas}})$, which both have image in G-equivariant objects, i.e. gaseous G-representations. These correspond to the maximal and minimal globalizations, which we'll discuss more in a couple of weeks.