Twistor $\mathbb{P}^1$ as the archimedean Fargues–Fontaine curve

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Our goal for this talk is to introduce the twistor $\mathbb{P}^1$, and argue that this is a good analogue at the archimedean place of the Fargues–Fontaine curve at p. Several key desiderata for such an analogue with respect to geometrizing the local Langlands correspondence will be discussed in future lectures, e.g. the Weil group action, L-parameters, and twistor and Hodge structures, but we can study some first properties which we'll see agree with what we'd want. Importantly, just as for the p-adic Fargues–Fontaine curve the twistor $\mathbb{P}^1$ should better be understood as relative to a test object A, i.e. a family of twistor $\mathbb{P}^1$'s; to understand this we'll need to introduce a good category of test spaces, which give archimedean analogues of perfectoid (or nil-perfectoid) rings.

1. The absolute twistor $\mathbb{P}^1$

By the twistor $\mathbb{P}^1$, we mean the inner form $X_{\mathbb{R}}$ of $\mathbb{P}^1_{\mathbb{R}}$ given by descending the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ along $z \mapsto -\frac{1}{\bar{z}}$. We enumerate some of its properties:

(i) The curve $X_{\mathbb{R}}$ has no real points: the residue field at each closed point is isomorphic to $\mathbb{C}$.

(ii) The points $0, \infty \in \mathbb{P}^1_{\mathbb{C}}$ glue to a ($\mathbb{C}$-valued) point $\infty \in X_{\mathbb{R}}$, which we treat as a distinguished point.

(iii) The automorphism group of $X_{\mathbb{R}}$ is $\mathbb{H}^\times/\mathbb{R}^\times$, with $\mathrm{O}(2)$ the stabilizer of ∞, acting on the residue field by the component map $\mathrm{O}(2) \to \mathbb{Z}/2 \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

(iv) The vector bundles on $X_{\mathbb{R}}$ decompose as direct sums of stable vector bundles, which are classified by their slopes $\lambda \in \frac{1}{2}\mathbb{Z}$ and either rank 1 (for integer λ) or rank 2 (for non-integer λ). If we write $\nu : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{R}}$ for the natural cover, we have $\nu^*\mathcal{O}_{X_{\mathbb{R}}}(1) \simeq \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)$ and $\nu_*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1) \simeq \mathcal{O}_{X_{\mathbb{R}}}(1/2)$.

We compare to some properties of the (p-adic) Fargues–Fontaine curve $X_{\mathbb{Q}_p,C}$ (for a fixed algebraically closed nonarchimedean field C):

(a) The residue fields of closed points in $X_{\mathbb{Q}_p,C}$ are given by algebraically closed nonarchimedean fields (namely untilts of $C^\flat$).

(b) There is a distinguished C-point x_C of $X_{\mathbb{Q}_p,C}$, such that $X_{\mathbb{Q}_p,C} \setminus \{x_C\}$ is an affine curve.

(c) The vector bundles on $X_{\mathbb{Q}_p,C}$ decompose as direct sums of stable vector bundles, which are classified by their slopes $\lambda \in \mathbb{Q}$. If $E/\mathbb{Q}_p$ is an extension of degree d and $\nu : X_{E,C} \to X_{\mathbb{Q}_p,C}$ the induced map, then $\nu^*\mathcal{O}_{X_{\mathbb{Q}_p,C}}(1) \simeq \mathcal{O}_{X_{E,C}}(d)$ and $\nu_*\mathcal{O}_{X_{E,C}}(1) \simeq \mathcal{O}_{X_{\mathbb{Q}_p,C}}(1/d)$.

In particular let's think briefly about alternatives to our definition of $X_{\mathbb{R}}$. The splitting behavior suggests that we should be looking at something like the projective line, so the most natural alternate choice would be just $\mathbb{P}^1_{\mathbb{R}}$. Here point (i) fails: $\mathbb{P}^1_{\mathbb{R}}$ has many real points.

More problematically point (iv) (compare (c) in the p-adic case) fails: the indecomposable bundles are all line bundles, but since we have a degree 2 extension $\mathbb{C}/\mathbb{R}$ we should expect a parallel to the Fargues–Fontaine curve to have indecomposable rank 2 bundles such as $\mathcal{O}(1/2)$.

There are two further properties we can study, which are slightly more complicated and which we'll have more to say about in a few weeks:

(v) There is an equivalence of categories between U(1)-equivariant semistable vector bundles on $X_{\mathbb{R}}$ ("U(1)-equivariant twistor structures") and pure $\mathbb{R}$-Hodge structures.

(vi) For any linear group G over $\mathbb{R}$, the set of isomorphism classes of G-torsors on $X_{\mathbb{R}}$ is in bijection with Kottwitz's set $B(\mathbb{R}, G)$.

Point (v) is analogous to the parametrization of p-adic Hodge structures via the Fargues–Fontaine curve, and we'll discuss it much more in a few weeks. Point (vi) is directly analogous to the parametrization of G-torsors on the Fargues–Fontaine curve; in the p-adic case, Kottwitz's set can be described in terms of isocrystals, and I hope to talk more about the archimedean analogue. By translating Hodge structures into twistor structures, one can translate Shimura data into the language of local Shimura data (G, b, μ) for a suitable group G, a minuscule cocharacter μ, and a basic element $b \in B(\mathbb{R}, G)$; it is very interesting to ask about the corresponding local Shimura variety, its relationship with the global Shimura variety, and how this might generalize the classical case. In the p-adic case local Shimura varieties exist under much weaker hypotheses than global ones, and it seems that this should be true here as well, but it is so far unclear how this generality might globalize. In any case we'll have much more to say about (vi) when we talk about Bun_G near the end of the semester.

Recall from our general framework that our goal for a reductive group G over $\mathbb{R}$ is to define a stack Bun_G, which should parametrize G-torsors on $X_{\mathbb{R}}$ in a suitable sense, whose derived category enlarges $D(*/G(\mathbb{R})^{\mathrm{la}})$ via an open embedding

$$*/G(\mathbb{R})^{\mathrm{la}} \hookrightarrow \mathrm{Bun}_G,$$

with other analogous loci for inner forms of G. In particular the trivial G-torsor on $X_{\mathbb{R}}$ should have automorphism group $G(\mathbb{R})^{\mathrm{la}}$.

(It's worth mentioning here that this means that the category of sheaves we'll study on Bun_G is just quasicoherent sheaves, whereas in the p-adic case we use (a version of) ℓ-adic sheaves or in the geometric setting we might use D-modules. The difference can be resolved by viewing our Bun_G as the transmutation of a more direct analogue Bun'_G to that of Fargues–Scholze, so that quasicoherent sheaves on Bun_G can be understood as something like D-modules or étale sheaves on Bun'_G.)

For $G = \mathbb{G}_{\mathrm{a}}$, the trivial $\mathbb{G}_{\mathrm{a}}(\mathbb{R})^{\mathrm{la}} = \mathbb{R}^{\mathrm{la}}$-torsor on any analytic stack X has automorphism group simply the global sections of the structure sheaf of X (as an abelian group). This is then supposed to be isomorphic to $\mathbb{R}^{\mathrm{la}}$. To see what this means, we need to think of each side as a functor: $\mathbb{R}^{\mathrm{la}}$ is already an analytic stack, sending an analytic ring A to $\mathbb{R}^{\mathrm{la}}(A)$, while the global sections of $X_{\mathbb{R}}$ means the functor sending A to the global sections of the *relative* curve $X_{\mathbb{R},A}$.

The most naive thing is to set $X_{\mathbb{R},A} = X_{\mathbb{R}} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,A$. But then the global sections will be A, and so this functor is represented by the *algebraic* affine line. More generally for this version we would get the algebraic group G as the automorphism group of the trivial G-torsor, rather than the analytic group $G(\mathbb{R})^{\mathrm{la}}$.

In fact, this problem already occurs in the p-adic world: we need the global sections of $X_{\mathbb{Q}_p}$ to be $\mathbb{Q}_p$ rather than $\mathbb{A}^1$. There, the solution is by using a less naive notion of a relative Fargues–Fontaine curve: for suitable test objects, namely perfectoid spaces in characteristic p, the relative Fargues–Fontaine curve $X_{\mathbb{Q}_p,S}$ is defined using the p-adic geometry of S, and does not agree with $X_{\mathbb{Q}_p} \times S$ (at least naively).

This suggests a two-part solution to our problem: first, we need to find a category of test objects replacing characteristic p perfectoid spaces in the archimedean setting; and second, after restricting to these we need to define families of archimedean Fargues–Fontaine curves in a more sophisticated way. Once this is accomplished, we can then try to check if our definition satisfies the above desideratum, i.e. the "real Banach–Colmez space" sending a test object A to the global sections of its relative archimedean Fargues–Fontaine curve is given by $\mathbb{R}^{\mathrm{la}}$.

2. Test categories of $\mathbb{R}_{\mathrm{gas}}$-algebras

Let A be a gaseous $\mathbb{R}$-algebra. Our goal is to define a chain of subsets

$$\mathrm{Nil}^{\dagger}(A) \subset A^{\circ\circ} \subset A^{\circ} \subset A^{\mathrm{bd}} \subset A$$

satisfying various properties; we can then use these to define certain properties of gaseous $\mathbb{R}$-algebras. For example, we'll say A is bounded if $A^{\mathrm{bd}} = A$.

The most straightforward approach is to work via a norm map: indeed, for each $f \in A(*)$ defines a map $|f| : \mathrm{AnSpec}\,A \to [0,\infty]$, and we can say f is bounded if this map takes image in $[0,\infty)$, or has norm 0 if it has image in $\{0\}$, etc. The bounded elements give a subring, and the norm 0 elements an ideal of this ring. Unfortunately this definition doesn't directly extend to $A(S)$ for arbitrary light profinite sets S. Our first goal is to give such an extension.

We fix a light profinite set S. The functor $A \mapsto A(S)$ can be viewed as the "S-dimensional affine space" $\mathbb{A}^S_{\mathbb{R}}$ over $\mathbb{R}_{\mathrm{gas}}$, and is represented by $\mathrm{AnSpec}\,\mathbb{R}[\mathbb{N}[S]]_{\mathrm{gas}}$. We will define subfunctors

$$\mathbb{A}^{S,\dagger}_{\mathbb{R}} \subset \mathbb{A}^{S,\circ\circ}_{\mathbb{R}} \subset \mathbb{A}^{S,\circ}_{\mathbb{R}} \subset \mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}} \subset \mathbb{A}^{S}_{\mathbb{R}}.$$

In the case $S = *$, these are the preimages under the norm map of

$$\{0\} \subset [0,1) \subset [0,1] \subset [0,\infty) \subset [0,\infty].$$

We can write $\mathbb{R}[\mathbb{N}[*]]_{\mathrm{gas}} = \mathbb{R}[T]_{\mathrm{gas}}$. Recall we have a method of forcing the variable T to be topologically nilpotent: we set $\mathbb{R}[\widehat{T}]_{\mathrm{gas}} = \mathbb{R}[\mathbb{N} \cup \{\infty\}]/(\infty)$, so rather than just parametrizing sequences $\{T^i\}$ we now parametrize sequences converging to 0. In a similar way by forming the one-point compactification $\mathbb{N}[S] \cup \{\infty\}$ of each $\mathbb{N}[S]$ we can force the generators to be topologically nilpotent, leading to the algebra $\mathbb{R}[\widehat{\mathbb{N}[S]}] = \mathbb{R}[\mathbb{N}[S] \cup \{\infty\}]/(\infty)$, whose spectrum we think of as a version of the S-dimensional unit disk

$$\mathbb{D}^S_{\mathbb{R}} = \mathrm{AnSpec}\,\mathbb{R}[\widehat{\mathbb{N}[S]}] \to \mathrm{AnSpec}\,\mathbb{R}[\mathbb{N}[S]] = \mathbb{A}^S_{\mathbb{R}}.$$

We have a scaling action of $\lambda \in \mathbb{R}$ on $\mathbb{A}^S_{\mathbb{R}}$, which for $|\lambda| \leq 1$ preserves the subspace $\mathbb{D}^S_{\mathbb{R}}$ and in general takes it to a disk of radius λ. One can then define the following subspaces of $\mathbb{A}^S_{\mathbb{R}}$:

(i) the intersection of all positive-radius disks

$$\mathbb{A}^{S,\dagger}_{\mathbb{R}} = \lim_{\lambda > 0} \lambda \mathbb{D}^S_{\mathbb{R}},$$

which we think of as a disk of infinitesimal radius;

(ii) the union of all disks of radius less than 1

$$\mathbb{A}^{S,\circ\circ}_{\mathbb{R}} = \operatorname*{colim}_{0<\lambda<1} \lambda \mathbb{D}^S_{\mathbb{R}},$$

which we think of as the open disk of radius 1;

(iii) the intersection of all disks of radius greater than 1

$$\mathbb{A}^{S,\circ}_{\mathbb{R}} = \lim_{\lambda > 1} \lambda \mathbb{D}^S_{\mathbb{R}},$$

which we think of as the overconvergent disk of radius 1;

(iv) the union of all positive-radius disks

$$\mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}} = \lim_{0<\lambda<\infty} \lambda \mathbb{D}^S_{\mathbb{R}}.$$

To check that this actually gives rise to well-defined subspaces of $\mathbb{A}^S_{\mathbb{R}}$ requires checking that the corresponding rings are idempotent $\mathbb{R}[\mathbb{N}[S]]_{\mathrm{gas}}$-algebras, which is a computation yet to be written down.

We have been thinking of these spaces as functors on gaseous $\mathbb{R}$-algebras, for each fixed light profinite set S. We now reverse our point of view: given a gaseous $\mathbb{R}$-algebra A, we can view each of these spaces as sending a light profinite set S to $\mathbb{A}^{S,?}_{\mathbb{R}}(A)$, giving condensed objects in the appropriate category. By studying the structures on the various affine space objects, we find the appropriate structures for these condensed objects: $\mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}}$ is a subring object of $\mathbb{A}^S_{\mathbb{R}}$, $\mathbb{A}^{S,\circ}_{\mathbb{R}}$ is a sub-multiplicative monoid, and $\mathbb{A}^{S,\dagger}_{\mathbb{R}}$ is an ideal object in the ring object $\mathbb{A}^S_{\mathbb{R}}$ or in fact $\mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}}$. We can therefore define our chain of subsets

$$\mathrm{Nil}^\dagger(A) \subset A^{\circ\circ} \subset A^\circ \subset A^{\mathrm{bd}} \subset A$$

sending a light profinite set S to

$$\mathbb{A}^{S,\dagger}_{\mathbb{R}}(A) \subset \mathbb{A}^{S,\circ\circ}_{\mathbb{R}}(A) \subset \mathbb{A}^{S,\circ}_{\mathbb{R}}(A) \subset \mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}}(A) \subset \mathbb{A}^S_{\mathbb{R}}(A),$$

such that $\mathbb{A}^{\mathrm{bd}} \subset A$ is a condensed (and in fact gaseous) subalgebra and $\mathrm{Nil}^\dagger(A) \subset A^{\mathrm{bd}}$ is a gaseous ideal. One can also formulate properties of $A^{\circ\circ}$ and A°, but they are more complicated and less relevant for us. Notably though the ring map $A^{\mathrm{bd}} \to A$ induces isomorphisms on each of the subspaces $\mathrm{Nil}^\dagger(-)$, $-^{\circ\circ}$, $-^\circ$, and $-^{\mathrm{bd}}$, so in particular $-^{\mathrm{bd}}$ is idempotent.

We say that A is bounded if $A^{\text{bd}} = A$. In this case, we say that its †-reduction is $A^{\dagger\text{-red}} = A/\operatorname{Nil}^\dagger(A)$, and that A is †-reduced if $A = A^{\dagger\text{-red}}$, or equivalently if $\operatorname{Nil}^\dagger(A) = 0$.

We can now use these structures to define our test objects. The idea is something like this: the simplest test spaces we could ask for are $\operatorname{Cont}(S, \mathbb{C})$ for light profinite sets S. For these for example we could hope to define families of twistor $\mathbb{P}^1$'s simply by base change from the absolute case. However this kind of space is not enough for our purposes: we need some larger test category in order to recover a good category of stacks. We will broaden it by allowing objects which are infinitesimally close to this kind of space, i.e. recover these spaces after †-reduction. In fact we'll allow ourselves to be even a little more general for convenience, though we'll see the difference doesn't matter too much.

We proceed a little more carefully. We say that a gaseous animated $\mathbb{C}$-algebra A (note that we're now working over $\mathbb{C}$) is totally disconnected if it is bounded and for each $s \in \pi_0 A^\triangleright(*)$, the resulting bounded $\mathbb{C}$-algebra

$$A_s = \operatorname*{colim}_{U \ni s} A(U)$$

has †-reduction $A_s^{\dagger\text{-red}} \simeq \mathbb{C}$. Globally, this means that for a totally disconnected $\mathbb{C}$-algebra A, we get a profinite set

$$S = \pi_0 A^\triangleright(*) = \operatorname{Hom}(A, \mathbb{C})$$

and a map $A \to \mathbb{C}^S$ whose kernel is $\operatorname{Nil}^\dagger(A)$. In fact the map $A \to \mathbb{C}^S$ factors as a map of condensed $\mathbb{C}$-algebras through $\operatorname{Cont}(S, \mathbb{C})$, yielding an injection

$$A^{\dagger\text{-red}} \to \operatorname{Cont}(S, \mathbb{C}).$$

We say that A is strongly totally disconnected if this map is an isomorphism, or equivalently if $A \to \operatorname{Cont}(S, \mathbb{C})$ is surjective. This is the †-deformation of $\operatorname{Cont}(S, \mathbb{C})$ we discussed above; strongly totally disconnected $\mathbb{C}$-algebras are also referred to in some places as nil-perfectoids, and fulfill a similar role to perfectoid spaces in p-adic geometry.

The difference between totally disconnected and strongly totally disconnected spaces is not too significant:

Proposition 1. *Let A be a totally disconnected $\mathbb{C}$-algebra such that $S = \operatorname{Hom}(A, \mathbb{C})$ is light. Then there is a descendable map $A \to \widetilde{A}$ with $\widetilde{A}$ strongly totally disconnected and the induced map $S \to \widetilde{S} = \operatorname{Hom}(\widetilde{A}, \mathbb{C})$ an isomorphism. In particular each tensor product $\widetilde{A} \otimes_A \cdots \otimes_A \widetilde{A}$ is strongly totally disconnected.*

Much like in the semiperfectoid case, the tensor products involved here are generally very hard to compute and may not be concentrated in degree 0 even when all rings involved are. Nevertheless we can generally avoid working with them so this doesn't present many issues in practice.

Proof sketch. For A totally disconnected, we have a map $A \to \operatorname{Cont}(S, \mathbb{C})$ where $S = \operatorname{Hom}(A, \mathbb{C})$. The claim is that we can reduce to the case where $A = \operatorname{colim}_i \operatorname{Cont}(S_i, \mathbb{C})$ for $S = \lim_i S_i$ a presentation by finite sets. In this case each $\operatorname{Cont}(S_i, \mathbb{C}) \to \operatorname{Cont}(S, \mathbb{C})$ splits and so the sequential limit is descendable.

The reduction to this case is supposed to be via base change: $A \mapsto S = \operatorname{Hom}(A, \mathbb{C})$ takes colimits to limits, so the claim is stable under base change. It is not however clear to me why every A is the base change of something of this form... □

In particular every totally disconnected $\mathbb{C}$-algebra has a cover by a strongly totally disconnected $\mathbb{C}$-algebra, and so we can mostly restrict to the latter (we'll say more about this in the next section). These two classes of rings will give us our test category. Before we say more about the general theory of stacks on this category, let's observe that we can now define the twistor $\mathbb{P}^1$ in families.

Let A be a totally disconnected $\mathbb{C}$-algebra, with $S = \mathrm{Hom}(A, \mathbb{C})$. Recall the absolute twistor $\mathbb{P}^1$ from the previous section, which we write as $X_{\mathbb{R}}$. We define the relative curve to be the pushout

$$\begin{array}{ccc} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})) & \xrightarrow{\infty} & X_{\mathbb{R}} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathrm{AnSpec}\,A & \longrightarrow & X_{\mathbb{R},A} \end{array}.$$

We make a few observations about this definition:

- The top map is given by the base change of $\infty : \mathrm{AnSpec}\,\mathbb{C} \to X_{\mathbb{R}}$ to $\mathrm{AnSpec}\,\mathrm{Cont}(S, \mathbb{C})$. In particular we are using that $\mathrm{Cont}(S, \mathbb{C})$ descends canonically to $\mathrm{Cont}(S, \mathbb{R})$ over $\mathbb{R}$, which of course is not true for all $\mathbb{C}$-algebras; this partially motivates the category of test objects.

- If $A \simeq \mathrm{Cont}(S, \mathbb{C})$, i.e. with trivial †-deformation, then the left vertical map is an isomorphism and so $X_{\mathbb{R},\mathbb{A}} \simeq X_{\mathbb{R}} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{Cont}(S, \mathbb{R})$ is just the base change of $X_{\mathbb{R}}$ to (the descended version of) A. In particular for $A = \mathbb{C}$ we recover the absolute curve $X_{\mathbb{R},\mathbb{C}} \simeq X_{\mathbb{R}}$.

- More generally, away from ∞ the relative curve is just the base change to $\mathrm{Cont}(S, \mathbb{R})$; in particular this does not actually depend on A but only on $S = \pi_0 A^{\triangleright}(*)$. The data of A itself gives the gluing data at the point at infinity.

- At least if A is strongly totally disconnected, this gives rise to a universal property for $X_{\mathbb{R},A}$: one can define an ∞-category of "abstract families of twistor $\mathbb{P}^1$'s" as profinite sets S together with †-thickenings of $X_{\mathbb{R}} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S, \mathbb{R})$. Taking the fiber at ∞ gives a functor to strongly totally disconnected spaces, of which $\mathrm{AnSpec}\,A \mapsto X_{\mathbb{R},A}$ is the left adjoint; so $X_{\mathbb{R},A}$ is the universal †-thickening of $X_{\mathbb{R}} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S, \mathbb{R})$ with fiber at ∞ given by $\mathrm{AnSpec}\,A$.

3. Stacks on totally disconnected rings

We can now define families of twistor $\mathbb{P}^1$'s as desired, but only relative to certain special test objects, namely totally disconnected $\mathbb{C}$-algebras. This is not a shortcoming of the theory: indeed it is exactly what we should expect parallel to the p-adic case, where our test objects are (covered by) totally disconnected affinoid perfectoid spaces. It does however present a technical problem: the functor sending A to e.g. the global sections or space of G-torsors on $X_{\mathbb{R},A}$ is, at least naively, not an analytic stack, since it is only defined on this subcategory of analytic rings. We will see that this does not present a major issue, but let's first make precise what this sort of object is: we define TotDisc to be the ∞-category of countably presented totally disconnected $\mathbb{C}$-algebras, and TDStack the category of functors

$$X : \mathrm{TotDisc} \to \mathrm{Ani}$$

commuting with finite products such that for every !-hypercover $\mathrm{AnSpec}\, A_\bullet \to \mathrm{AnSpec}\, A$ satisfying !-descent, the map

$$X(A) \to \lim X(A_\bullet)$$

is an isomorphism. (The restriction to countably presented algebras A is to ensure that the resulting profinite set $S = \mathrm{Hom}(A, \mathbb{C})$ is light.)

For $A \in \mathrm{TotDisc}$, we get a corresponding object $\mathrm{TDSpec}(A) \in \mathrm{TDStack}$ sending $B \mapsto \mathrm{Hom}(A, B)$.

Since A is also an analytic ring, we could instead take its analytic spectrum $\mathrm{AnSpec}\, A$, giving an analytic stack. This operation gives the pullback of a map of ∞-topoi $\mathbf{AnStack}_{\mathbb{C}_{\mathrm{gas}}} \to \mathrm{TDStack}$, which more generally assigns to any object of TDStack an analytic stack over $\mathbb{C}_{\mathrm{gas}}$. We sometimes refer to this as the analytic realization. In particular, via this construction we can recover analytic stacks from objects defined a priori only on totally disconnected $\mathbb{C}$-algebras, which is a good sign for problems defined over our relative Fargues–Fontaine curve.

Since we have observed that every totally disconnected $\mathbb{C}$-algebra has a cover by a strongly totally disconnected one, we might hope that we can restrict to the subcategory $\mathrm{StrTotDisc} \subset \mathrm{TotDisc}$ of strongly totally disconnected $\mathbb{C}$-algebras. This is true up to size issues: the corresponding strongly totally disconnected algebras are not necessarily countably presented. However any totally disconnected A is the $\aleph_1$-filtered colimit of countably presented totally disconnected A_i, so we can extend a stack $X \in \mathrm{TDStack}$ to all totally disconnected A by

$$X(A) := \operatorname*{colim}_i X(A_i).$$

Then using this definition objects of TDStack are determined by their restriction to strongly totally disconnected $\mathbb{C}$-algebras. To go the other way, given a moduli problem on strongly totally disconnected algebras it suffices to check that it commutes with $\aleph_1$-filtered colimits to get an object of TDStack.

It remains to see that our test category is large enough to give covers of suitable analytic stacks. This follows from the following proposition, applied to overconvergent algebras on compact Stein spaces:

Proposition 2. *For A a countably presented bounded gaseous $\mathbb{C}$-algebra, assume that there exists a finite-dimensional metrizable compact Hausdorff space S and a map*

$$\mathrm{AnSpec}\, A \to S_{\mathrm{Betti}},$$

determined by idempotent A-algebras A_Z for each closed $Z \subset S$, such that each point has a neighborhood including a connective and bounded closed subset $Z \subset S$, and for each $s \in S$ the stalk A_s satisfies $A_s^{\dagger\text{-red}} \simeq \mathbb{C}$. Then for any light profinite cover $\widetilde{S} \to S$ the fiber product

$$\mathrm{AnSpec}\, A \times_{S_{\mathrm{Betti}}} \widetilde{S}_{\mathrm{Betti}}$$

is isomorphic to $\mathrm{AnSpec}\, \widetilde{A}$ for some totally disconnected algebra A, and the map $A \to \widetilde{A}$ is descendable. Further each tensor product $\widetilde{A} \otimes_A \cdots \otimes_A \widetilde{A}$ is totally disconnected.

The proof is relatively straightforward from the hypotheses, and we omit it for time.

In particular, for A satisfying the conditions of the proposition, we get a functor on TotDisc sending $B \mapsto \mathrm{Hom}(A, B)$. If A were itself totally disconnected this would just be $\mathrm{TDSpec}(A)$, but in general such an object is not defined a priori, though we abusively denote it by $\mathrm{TDSpec}(A)$. Instead, we get an object of TDStack whose pullback to **AnStack** is $\mathrm{AnSpec}(A)$; indeed, we have a cover $\mathrm{TDSpec}(A) \to \mathrm{TDSpec}(\widetilde{A})$ with $\widetilde{A}$ as in the proposition, and the Čech nerve agrees in analytic rings and totally disconnected algebras. This is the pushforward of $\mathrm{AnSpec}\, A$ along the map of topoi **AnStack** $\to$ TDStack.

We mention some other examples:

- The functor on totally disconnected algebras sending $A \mapsto A(*)$ realizes to the analytic stack $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}}$. Indeed, since each totally disconnected algebra is bounded the realization maps into the analytic affine line, and after restriction to compact Stein spaces this map is an isomorphism by the observation above that $\mathrm{TDSpec}(A)$ realizes to $\mathrm{AnSpec}(A)$ for A the overconvergent algebra of compact Stein spaces.

  Notably if we studied the same functor on analytic rings we would obtain the algebraic affine line $\mathbb{A}^1_{\mathbb{C}_{\mathrm{gas}}}$ instead!

- The functor on totally disconnected algebras sending $A \mapsto (\mathrm{Nil}^\dagger(A))(*)$ realizes to $\mathbb{A}^{1,\dagger}_{\mathbb{C}_{\mathrm{gas}}}$, as it is the TDSpec of the ring of germs of holomorphic functions at 0.

Finally, we can reinterpret the Betti functor in terms of TDStack: it is the pullback along a morphism of topoi $\pi : \mathrm{TDStack} \to \mathrm{Cond}(\mathrm{Ani})$, taking a light profinite set S to

$$\pi^* S = \mathrm{TDSpec}(\mathrm{LocConst}(S, \mathbb{C}))$$

which realizes to the analytic stack S_{Betti}. In the context of totally disconnected stacks, we also have the functor $A \mapsto \mathrm{Hom}(A, \mathbb{C})$, commuting with finite limits and covers and so defining another morphism of topoi $\psi : \mathrm{Cond}(\mathrm{Ani}) \to \mathrm{TDStack}$, which is a section of π with $\psi_* = \pi^*$.

For $X \in \mathrm{TDStack}$, the unit transformation gives a map $X \to \pi^* \psi^* X = (\psi^* X)_{\mathrm{Betti}}$, so we can think of $\psi^* X$ as the "underlying condensed anima." This is the sheafification of the functor $S \mapsto X(\mathrm{Cont}(S, \mathbb{C}))$ (note this is a very transmutation-like formula!). Similarly, for any condensed anima X the functor $A \mapsto X(\mathrm{Hom}(A, \mathbb{C}))$ gives an object of TDStack, whose analytic realization recovers X_{Betti}, as this is the composite pullback **AnStack** $\to$ TDStack $\to$ Cond(Ani).

We can now reinterpret analytic Riemann–Hilbert as the following statement.

Proposition 3. *There is a short exact sequence of sheaves on* TotDisc *sending strongly totally disconnected A to*

$$0 \to \mathrm{Nil}^\dagger(A) \to A \to \mathrm{Cont}(\mathrm{Hom}(A, \mathbb{C}), \mathbb{C}) \to 0,$$

which realizes to the exact sequence in $\mathbf{AnStack}_{\mathbb{C}_{\mathrm{gas}}}$

$$0 \to \mathbb{A}^{1,\dagger}_{\mathbb{C}_{\mathrm{gas}}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}} \to \mathbb{C}_{\mathrm{Betti}} \to 0.$$

Proof. For totally disconnected A, we have an injective map $A \to \mathrm{Cont}(\mathrm{Hom}(A, \mathbb{C}), \mathbb{C})$ with kernel $\mathrm{Nil}^\dagger(A)$, so it suffices to show that this map becomes an isomorphism after sheafification. We know that it is an isomorphism for strongly totally disconnected algebras and that any A has a cover by such an algebra, so this holds; the final statement follows from our discussion of the definitions of these stacks. □

Now Riemann–Hilbert for $\mathbb{A}^1$ is just the identification of $\mathbb{A}^{1,\mathrm{an}}/\mathbb{A}^{1,\dagger} \simeq \mathbb{C}_{\mathrm{Betti}}$, and in general follows by transmutation.

4. Real Banach–Colmez spaces

Recall our main goal in all this was to find a version of the twistor $\mathbb{P}^1$ which works well in families, in the sense that its global sections functor should recover $\mathbb{R}^{\mathrm{la}}$ rather than $\mathbb{A}^1$ or similar. We now have a definition in families, so the question is whether we've achieved our goal.

More generally, we can study the following archimedean analogue of Banach–Colmez spaces: fix a coherent sheaf M on $X_{\mathbb{R}}$, and consider the object of TDStack

$$A \mapsto \Gamma(X_{\mathbb{R},A}, M|_{X_{\mathbb{R},A}}).$$

This is the Banach–Colmez space $\mathcal{BC}(M)$. It is valued in $D_{\geq 0}(\mathbb{R}_{\mathrm{gas}})$, but we forget the condensed structure to view it as an animated $\mathbb{R}$-vector space object in TDStack.

Proposition 4. *The realization of $\mathcal{BC}(\mathcal{O}_{X_{\mathbb{R}}})$ as an analytic stack over $\mathbb{C}_{\mathrm{gas}}$ is $\mathbb{R}^{\mathrm{la}}$.*

Proof. If A is strongly totally disconnected with $S = \mathrm{Hom}(A, \mathbb{C})$, then $\Gamma(X_{\mathbb{R},A}, \mathcal{O})$ is the fiber product over $\mathrm{Cont}(S, \mathbb{C})$ of A with

$$\Gamma(X_{\mathbb{R}} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S, \mathbb{R}), \mathcal{O}) = \mathrm{Cont}(S, \mathbb{R}).$$

This can be understood as the "real part" of A under the map $A \to \mathrm{Cont}(S, \mathbb{C}) = A^{\dagger\text{-red}}$; in other words if we write $\mathrm{Im} : \mathrm{Cont}(S, \mathbb{C}) \to \mathrm{Cont}(S, \mathbb{R})$ for the imaginary part, then $\mathcal{BC}(\mathcal{O}_{X_{\mathbb{R}}})(A)$ is the kernel of the composite map

$$A \to A^{\dagger\text{-red}} \simeq \mathrm{Cont}(S, \mathbb{C}) \xrightarrow{\mathrm{Im}} \mathrm{Cont}(S, \mathbb{R}).$$

Taking realizations, this becomes

$$\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}} \to \mathbb{C}_{\mathrm{Betti}} \xrightarrow{\mathrm{Im}} \mathbb{R}_{\mathrm{Betti}}.$$

The kernel of this map is in turn

$$\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}} \times_{\mathbb{C}_{\mathrm{Betti}}} \mathbb{R}_{\mathrm{Betti}},$$

which is the real line inside of $\mathbb{C}$ viewed as a complex manifold, i.e. $\mathbb{R}$ viewed as a real-analytic manifold. But this is just $\mathbb{R}^{\mathrm{la}}$. □