Real local Langlands geometrization: introduction

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1. Introduction

The goal of this seminar is to understand Scholze's recent work on the geometrization of the local Langlands program at archimedean places. This is parallel to his work with Fargues at nonarchimedean places, but requires new machinery coming from his work with Clausen on condensed mathematics and analytic stacks. Speculatively, it is potentially the next step in the road towards a geometrization of the *global* Langlands program, which should be compatible with local Langlands at all places and so requires theories at finite and infinite places that can be treated more or less uniformly.

We start with a very brief discussion of what we mean by the Langlands program. There are roughly four settings in which the Langlands program can be discussed (as well as a large number of generalizations and analogues which we will not get into), which we represent by the following diagram:

$$\begin{tikzcd}
& \text{Global arithmetic (number fields, function fields over } \mathbb{F}_q\text{)} \arrow[dl, "\text{completion}"'] \arrow[dr, "\text{base change}"] & \\
\text{Local arithmetic (}p\text{-adic fields, } \mathbb{R}, \mathbb{C}, \mathbb{F}_q((t))\text{)} \arrow[dr, "\text{base change}"'] & & \text{Global geometric (function fields over } k=\overline{k}\text{)} \arrow[dl, "\text{completion}"] \\
& \text{Local geometric (local function fields (} k((t))=\overline{k}((t))\text{))} &
\end{tikzcd}$$

At each level of the diagram on a field F (respectively global or local and arithmetic or geometric) and for a reductive group G over F, the Langlands program considers two objects: an "automorphic" object $\mathcal{A}_G(F)$ and a "Galois" object $\mathcal{B}_{\check{G}}(F)$, where $\check{G}$ is the Langlands dual group. At the global level, $\mathcal{A}_G(F)$ is a vector space of automorphic forms for G over F, while $\mathcal{B}_{\check{G}}(F)$ is a space of Galois representations $\mathrm{Gal}_F \to \check{G}$.[1] A famous example for $G = \mathrm{GL}_2 = \check{G}$ is the modularity theorem: suitable automorphic forms over $\mathbb{Q}$, corresponding to weight 2 modular forms, correspond to suitable Galois representations, given by the Tate modules of elliptic curves over $\mathbb{Q}$. This was proven by Wiles, Taylor–Wiles, and Breuil–Conrad–Diamond–Taylor.

At the local arithmetic or global geometric levels, $\mathcal{A}_G(F)$ and $\mathcal{B}_{\check{G}}(F)$ should be categories: in the local arithmetic case, $\mathcal{A}_G(F)$ is essentially the category of smooth $G(F)$-representations and $\mathcal{B}_{\check{G}}(F)$ representations of the Weil–Deligne group (i.e. L-parameters), a modification of Gal_F, while in the global geometric case $\mathcal{A}_G(F)$ is a category of sheaves (or D-modules or similar) on a stack Bun_G parametrizing G-bundles on the relevant curve, and $\mathcal{B}_{\check{G}}(F)$ is a category of quasi-coherent sheaves on a stack $\mathrm{Loc}_{\check{G}}$ parametrizing $\check{G}$-local

[1] In general we should replace $\check{G}$ with ${}^L G = \check{G} \rtimes \mathrm{Gal}_F$, but this is literally true for G split.

systems. (The local geometric setting involves 2-categories and is a little more complicated than I want to get into right now.) The Langlands program can be viewed as the statement $\mathcal{A}_G = \mathcal{B}_{\check{G}}$, uniformly across settings.

Making precise exactly what this should mean and what kinds of objects $\mathcal{A}_G$ and $\mathcal{B}_{\check{G}}$ should be is (part of) the goal of the relative Langlands correspondence, together with the implications of this formulation re functoriality. For our purposes, the most interesting observation from this formulation is that the local arithmetic setting is in a sense on the same footing as the global geometric setting. This led Fargues to speculate [2] that it should be possible to interpret local arithmetic Langlands at nonarchimedean places as geometric Langlands on an "exotic curve," the Fargues–Fontaine curve. As the geometric Langlands program is significantly better-developed than the arithmetic one (e.g. otherwise it is, to the best of my knowledge, not known how to give a general formulation of the local Langlands conjectures for all groups!), this would be very helpful, e.g. giving a categorical formulation.

This was achieved in Fargues's work with Scholze in 2021 [3], using large amounts of p-adic geometry. We will (very) briefly review their geometrization in the next section. However, this simultaneously raises the question of how to geometrize the archimedean case and makes clear why we should expect it to be difficult: the p-adic geometry developed by Scholze does not exist in the archimedean setting. Instead, we have the much more analytic theories of real and complex geometry, which are less amenable to algebraic methods.

Recent work of Clausen and Scholze however makes this problem more approachable. Via the methods of condensed mathematics [4], one can construct theories of analytic and complex geometry [5], [6] with well-behaved categorical properties and so which can be studied via the tools of algebraic geometry. Paralleling work of Rodríguez-Camargo in the p-adic setting, it turns out that one can then find analogues of some of the key p-adic constructions in [3], and build a similar picture in the archimedean setting.

2. The p-adic case

Let's first say a little more about how Fargues–Scholze's geometrization works. Let F be a p-adic field. On the automorphic side, we can think of smooth representations of $G(F)$ (say with $\overline{\mathbb{Q}}_\ell$-coefficients) as ℓ-adic sheaves on $*/G(F)$, or (passing to derived categories) $D([*/G(F)], \overline{\mathbb{Q}}_\ell)$. Making precise what we mean by this category is not trivial: if we take the limit of categories with torsion coefficients, we get undesirable behavior (e.g. everything is ℓ-adically complete, and if we take the category of condensed $\mathbb{Q}_\ell$-sheaves the resulting category is too big. Instead, we take a solid subcategory of ℓ-adic sheaves, and then cut out a lisse subcategory corresponding to smooth representations and recovering the good behavior of étale sheaves when we take torsion coefficients. This is a fairly mild application of condensed mathematics corresponding to the fairly mild presence of analytic information in the p-adic setting.

On the Galois side, we can construct a stack of L-parameters $Z_{F,\check{G}}$, which is in fact just a scheme over $\mathbb{Z}_\ell$. The local Langlands conjectures for G can then be formulated as a fully faithful functor $D_{\mathrm{lis}}([*/G(F)], \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{qc}}(Z_{F,\check{G}}/\check{G})$, with classical smooth representations on the left corresponding to skyscraper sheaves on the right supported at the $\check{G}$-conjugacy class of the associated L-parameter.

It is then natural to ask if we can upgrade this functor to an equivalence. Modulo certain restrictions on the Galois side (e.g. following the geometric Langlands program we should restrict to sheaves with nilpotent singular support and coherent cohomology), the answer is (conjecturally) yes: there is a stack Bun_G such that $[*/G(F)]$ embeds into Bun_G, and pushforward along this embedding gives an embedding $D([*/G(F)]) \to D(\mathrm{Bun}_G)$. The same holds for inner twists G_b of G, giving other strata of Bun_G; thus considering Bun_G instead of $*/G(F)$ corresponds to the strategy of studying the representation theory of all inner twists simultaneously. After finding the right derived category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ as above and imposing suitable restrictions on the Galois side, the embedding above should then upgrade to an equivalence $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \simeq D^{b,\mathrm{qc}}_{\mathrm{coh}}(Z_{F,\check{G}}/\check{G})$.

To complete the story, we need to say what Bun_G is. Via our slogan that nonarchimedean local Langlands should be geometric Langlands on the Fargues–Fontaine curve, what this means is that Bun_G should parametrize G-bundles on the Fargues–Fontaine curve: for each test object S, we associate to it a "curve" X_S, so $\mathrm{Bun}_G(S) = \{G\text{-bundles on } X_S\}$. The test objects S are characteristic p perfectoid spaces, and X_S is constructed such that its degree 1 divisors parametrize untilts of S modulo the Frobenius action. Equivalently, if Div^1 is the stack parametrizing degree 1 divisors on the Fargues–Fontaine curve, then $\mathrm{Div}^1 \simeq (\mathrm{Spd}\, \breve{F})/\varphi^{\mathbb{Z}}$.

Let me briefly try to make sense of the previous two sentences: there exist objects in p-adic geometry called perfectoid spaces, which can exist either in characteristic 0 or characteristic p, equipped with a tilting operation $X \mapsto X^\flat$ taking a perfectoid space to a characteristic p perfectoid space (which is the identity on characteristic p perfectoid spaces). An untilt of S is a perfectoid space X together with an isomorphism $X^\flat \simeq S$. We can bundle this data as a functor $S \mapsto \{(X, X^\flat \simeq S)\}$, which we call $\mathrm{Spd}\, \mathbb{Z}_p$ or $(\mathrm{Spa}\, \mathbb{Z}_p)^\diamondsuit$. More generally, we could additionally ask that X live over an adic ring R, in which case we call the functor $S \mapsto \{(X \to \mathrm{Spa}\, R, X^\flat \simeq S)\}$ $\mathrm{Spd}\, R$ or $(\mathrm{Spa}\, R)^\diamondsuit$. The Frobenius of S acts on $(\mathrm{Spd}\, R)(S)$ by $(X, X^\flat \simeq S) \mapsto (X, X^\flat \simeq S \xrightarrow{\varphi} S)$, since φ is an isomorphism.

We want to consider untilts differing by Frobenius to be equivalent. Suggestively, we observe that $\pi_1((\mathrm{Spec}\, \breve{F})/\varphi^{\mathbb{Z}})$ is exactly the Weil group W_F, the preimage of $\mathbb{Z} \subset \mathrm{Gal}_{\mathbb{F}_q}$ under the natural surjection $\mathrm{Gal}_F \to \mathrm{Gal}_{\mathbb{F}_q}$, so studying $\mathrm{Div}^1 = (\mathrm{Spd}\, \breve{F})/\varphi^{\mathbb{Z}}$ is already interesting. Defining Bun_G via the Fargues–Fontaine curve means that the Hecke stacks give correspondences between Bun_G and $\mathrm{Bun}_G \times \mathrm{Div}^1$, which via the W_F-action maps to $\mathrm{Bun}_G \times [*/W_F]$; thus Hecke operators give maps $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to D(\mathrm{Bun}_G \times [*/W_F], \overline{\mathbb{Q}}_\ell) \simeq D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_F}$, i.e. W_F-equivariant objects in $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$.

This shows us roughly what we need for an archimedean analogue: a suitable object $*/G(\mathbb{R})$ and a good category of sheaves on it; a stack of L-parameters for the Galois side; an object Bun_G admitting a suitable map from $*/G(\mathbb{R})$, together with a good category of sheaves on it; and in order to construct Bun_G, a replacement for the Fargues–Fontaine curve, and a related replacement of Div^1. Combining everything together, we should be able to write down a conjecture of the same shape as in the p-adic case.

3. Outline of the seminar

First of all, we'll want to review what the classical real local Langlands program says, concentrating on the case of GL_2 for concreteness; this will occupy our first talk next week (calling this the zeroth talk). The next two talks will be concerned with reviewing the theory of analytic rings and analytic stacks, which are the technical basis for most of what we'll do in the seminar. We'll generally avoid most proofs and technical details where possible, trying to just get enough definitions and intuition to work with these methods going forwards.

The core material of the seminar kicks off the following week with talk (4) on the Riemann–Hilbert correspondence. The classical Riemann–Hilbert correspondence relates regular holonomic D-modules to perverse constructible sheaves on a suitable space (say a complex manifold). Scholze's analytic Riemann–Hilbert correspondence passes through the theory of transmutation: the philosophy is that many different types of cohomologies or sheaf theories (de Rham cohomology and D-modules, prismatic/crystalline cohomology and crystals, etc.) should be viewed as quasi-coherent cohomology/sheaves on a "transmuted" stack: for example, D-modules on a complex manifold X are equivalent to quasi-coherent sheaves on its de Rham space X_{dR}. Scholze defines a version of the de Rham stack in the analytic setting $X_{\mathrm{dR}}^{\mathrm{an}}$ (following Rodríguez-Camargo [1]) as well as a "Betti stack" X_{Betti} whose transmutation corresponds to "Betti sheaves" on X. The analytic Riemann–Hilbert correspondence is then an identification $X_{\mathrm{dR}}^{\mathrm{an}} \simeq X_{\mathrm{Betti}}$, which on categories of quasi-coherent sheaves identifies D-modules with Betti sheaves; we can then view regular holonomic D-modules and perverse constructible sheaves respectively as subcategories of each side which are identified under this equivalence.

This will be important for our geometrization to translate between the kind of language appearing in Fargues–Scholze (where we're interested in categories more closely resembling Betti sheaves and subcategories thereof) and the kind of machinery arising in real local Langlands which tends to produce D-modules. In particular, we classically often study real representations through the theory of $(\mathfrak{g}, K)$-modules (translating which into the language of analytic stacks will largely occupy talk (5), and Beilinson–Bernstein localization (talk 6) lets us translate these in turn into D-modules on the flag variety.

Now that we can turn the real representation-theoretic language into stacky language, we turn to the meat of the geometrization program in this setting: talks (7) - (9) are occupied with the construction of archimedean analogues of Fargues–Fontaine curves (as families of twistor $\mathbb{P}^1$'s), Div^1, and the stack of L-parameters respectively. Talk (10) relates this theory to twistor structures and explains a notion of diamondization in this setting. Talk (11) combines the material of the previous sections and defines Bun_G to complete the geometrization, and in talk (12) we work out an example in the form of non-abelian real Lubin–Tate theory, recovering for example a version of Matsushima's formula.

References

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