# Classical real representation theory and local Langlands

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In this talk I am going to make a lot of unjustified statements, some of which are probably false; please feel free to correct me, and don't take anything I say too seriously. We'll return to some of this in future talks from a more "modern" viewpoint.

## 1. Real representation theory and $(\mathfrak{g}, K)$-modules

Let $G = G^{\mathrm{alg}}(\mathbb{R})$ be a real Lie group (we'll have $\mathrm{GL}_2(\mathbb{R})$ in mind). We want to study its representations, subject to suitable adjectives (smooth, admissible, ...). The study of (continuous) one-dimensional representations is, in general, easy: these factor through one-dimensional Lie groups, which (modulo some possible finite groups) are just $\mathbb{R}$ or $S^1$ whose characters are well-understood. For example, for $G = \mathrm{GL}_2(\mathbb{R})$ the characters all factor through the determinant map $\mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ and so are all given by either $g \mapsto |\det g|^s$ or $g \mapsto \mathrm{sign}(\det g)|\det g|^s$ for some complex number $s$.

There are also some natural finite-dimensional representations: most obviously the 2-dimensional standard representation Std and its symmetric powers, as well as their tensor products with characters as above. In fact these completely classify the finite-dimensional irreducible representations. Our main focus will be on infinite-dimensional representations, which are more complicated.

We have a variety of decompositions of real groups, e.g. $G = PK$ for a parabolic $P = MN$, so we can inductively form further representations by parabolic induction. For example in the $G = \mathrm{GL}_2(\mathbb{R})$ case we can take $M = \mathbb{R}^\times \times \mathbb{R}^\times$ the diagonal torus, $N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$, so $P = MN$ is the standard Borel subgroup of upper triangular matrices; for $\chi = \chi_1 \boxtimes \chi_2$ a character of $M$, we can take the parabolic induction

$$\mathrm{Ind}_P^G(\chi) = \{f : G \to \mathbb{C} | f(mng) = \chi(m)f(g)\}.$$

For this to make sense, we need to pin down what kind of functions we mean. There are a few options: we could take any functions at all, which gives a very large space; continuous or smooth functions, or even analytic functions; smooth or analytic distributions; etc. On the other hand we should only really have one theory of parabolic induction, so it is inconvenient to have so many options.

The standard workaround is to instead pass to the theory of $(\mathfrak{g}, K)$- or Harish-Chandra modules, which as we'll see is a much more algebraic theory and avoids these functional analytic difficulties. It comes at the cost however of no longer working directly with $G$-representations; for example it is harder to formulate an analogue to the $p$-adic situation, where no such algebraization is necessary. (This is largely due to the fact that $p$-adically we work with $\ell$-adic coefficients, while here we're working over a real field with real coefficients; the better analogue would be $p$-adic Langlands, i.e. with $p$-adic coefficients, where analytic difficulties do in fact arise.) The first hurdle we'll encounter in a few weeks is how to work directly with $G$-representations; this is where the condensed machinery is critical, allowing an algebraic treatment of the functional analysis (specifically giving a good derived category of locally analytic representations).

For the moment, however, we focus on the classical theory; we'll see later how to connect it to the locally analytic theory. Let's briefly recall the notion of a $(\mathfrak{g}, K)$-module.

Suppose $(\pi, V)$ is a representation of $G$. Its smooth vectors can be thought of as $v \in V$ such that $g \mapsto g \cdot v$ is a smooth map $G \to V$; these form a dense subspace $V^\infty \subset V$, which carries a natural action of $\mathfrak{g}$. For example on $V = L^2(G)$ the smooth vectors $V^\infty$ consist of $L^2$-functions on $G$ whose derivatives are all also $L^2$.

On the other hand, let $K$ be a maximal compact subgroup of $G$. For a representation $V$ of $K$, a vector $v \in V$ is $K$-finite if $K \cdot v = \{k \cdot v | k \in K\}$ is contained in a finite-dimensional subspace of $V$; we write $V^{(K)}$ for the $K$-finite vectors in $V$. Note that since $K$ is compact, its action on $V$ is a direct sum of finite-dimensional irreducible representations, on each of which it acts analytically, so the action on $V$ is analytic. If the $K$-action extends to a $G$-action on $V$, then $V^{(K)}$ is dense in $V$, and while it is not necessarily $G$-stable it is $\mathfrak{g}$-stable.

In particular, given a $G$-representation $V$ we can pass to the dense subspace of vectors which are both smooth and $K$-finite: $V \mapsto V^{\infty,(K)} = V^\infty \cap V^{(K)}$. By construction this has compatible well-behaved $\mathfrak{g}$- and $K$-actions. Abstracting these properties, we arrive at the notion of a $(\mathfrak{g}, K)$-module: this is a $\mathbb{C}$-vector space $V$ equipped with actions of $\mathfrak{g}$ and $K$, such that

- the $K$-action is locally finite and continuous (equivalently analytic);
- the differential of the $K$-action agrees with (the restriction of) the $\mathfrak{g}$-action;
- for $k \in K$, $X \in \mathfrak{g}$, we have
$$k \cdot (X \cdot v) = (\mathrm{Ad}(k) \cdot X) \cdot (k \cdot v).$$

Thus the above construction gives a functor from $G$-representations to $(\mathfrak{g}, K)$-modules; representations with the same $(\mathfrak{g}, K)$-module are said to be infinitesimally equivalent.

We say that a $G$-representation $V$ is admissible (or $K$-admissible) if the $K$-representation $V^{(K)}$ has finitely many factors of each irreducible $K$-representation. We can make the same definition for $(\mathfrak{g}, K)$-modules, and a $G$-representation is admissible if and only if its associated $(\mathfrak{g}, K)$-module is. We often restrict to admissible representations, which have the following nice property:

**Proposition** (Harish-Chandra)**.** *Suppose $V$ is an admissible $G$-representation with associated $(\mathfrak{g}, K)$-module $V^{\infty,(K)}$. There is a one-to-one correspondence between closed subrepresentations of $V$ and sub-$(\mathfrak{g}, K)$-modules. In particular $V$ is irreducible as a $G$-representation if and only if $V^{\infty,(K)}$ is an irreducible $(\mathfrak{g}, K)$-module.*

**Corollary.** *Schur's lemma holds for admissible representations, i.e. if $V$ is an admissible irreducible $G$-representation then $\mathrm{End}_G(V) \simeq \mathbb{C}$.*

Indeed, the functor above gives a map $\mathrm{End}_G(V) \to \mathrm{End}_{(\mathfrak{g},K)}(V^{\infty,(K)})$, which is injective by density, and $V^{\infty,(K)}$ is by assumption and the above result an irreducible $K$-representation of countable dimension and therefore has only scalar $K$-endomorphisms.

We now return to the question of parabolic induction. Let $P = MAN$ be a parabolic subgroup of $G$ (so for our example of $\mathrm{GL}_2$, $M$ is trivial, $A = \mathbb{R}^\times \times \mathbb{R}^\times$, and $N$ is as above; or slightly better $M = \{\pm 1\} \times \{\pm 1\}$, $A = \mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}$), with the torus $A$ acting on $N$ with weight $2\rho$. If $(\sigma, V)$ is a representation of $M$ and $\lambda \in \mathfrak{a}^* = (\mathrm{Lie}\, A)^*_{\mathbb{C}}$, then $\mathrm{Ind}_P^G(\sigma, \lambda) = \mathrm{Ind}_P^G(\sigma \boxtimes (\lambda + \rho) \boxtimes 1)$ is (the completion of) the space of continuous functions $f : G \to V$ such that

$$f(gman) = e^{-\langle \lambda + \rho, \log a \rangle} \sigma(m)^{-1} f(g).$$

Writing $\lambda(X) = \chi(\exp(X))$ for $X \in \mathfrak{a}$ we recover in the $\mathrm{GL}_2$-case the formula $f(gan) = \chi(a) f(g)$. More generally if $M$ is compact we can take $\sigma$ to be finite-dimensional irreducible and get something that looks fairly similar to what we expect from parabolic induction.

This gives better results for $(\mathfrak{g}, K)$-modules, for which purposes we look at $\mathrm{Ind}_P^G(\sigma, \lambda)^{\infty,(K)}$. In particular we have the following result towards classification, due to Casselman:

**Theorem.** *Any irreducible $(\mathfrak{g}, K)$-module appears as a submodule of some $\mathrm{Ind}_P^G(\sigma, \lambda)^{\infty,(K)}$ as above.*

There is a similar classification due to Langlands in terms of quotients instead: any irreducible $(\mathfrak{g}, K)$-module is the unique irreducible subquotient of the parabolic induction of a tempered representation. In fact, by adding a temperedness condition one can reduce to the induction of discrete series; in this sense the archimedean case is simpler than the nonarchimedean case in that there are no (other) supercuspidals.

Since we have passed from $G$-representations to $(\mathfrak{g}, K)$-modules, a remaining question is how to go in the reverse direction: can we find "globalizations" of $(\mathfrak{g}, K)$-modules, i.e. representations in their preimage under the above functor, in a natural way? Are they unique?

In fact there are (at least) two natural ways to go in the reverse direction. Abstractly, this is the statement that the functor $V \mapsto V^{\infty,(K)}$ from $G$-representations to $(\mathfrak{g}, K)$-modules has both a left adjoint $m$ and a right adjoint $M$, resulting in unit and counit maps

$$m(V^{\infty,(K)}) \to V \to M(V^{\infty,(K)})$$

for any $G$-representation $V$. Indeed since $V \mapsto V^{\infty,(K)}$ is faithful both maps are injective, so any $G$-representation $V$ lives between a minimal and maximal globalization of its $(\mathfrak{g}, K)$-module; in other words, for any $(\mathfrak{g}, K)$-module $W$ and globalization $V$, we have inclusions

$$m(W) \hookrightarrow V \hookrightarrow M(W),$$

justifying calling $m(W)$ and $M(W)$ the minimal and maximal globalizations generally. Explicitly, let $W^\vee = W^{*,(K)}$ be the admissible dual, i.e. the locally $K$-finite piece of the algebraic dual. Then

$$M(W) = \mathrm{Hom}_{(\mathfrak{g},K)}(W^\vee, C^\infty(G))$$

(we could replace smooth functions with analytic functions or distributions and get the same thing up to isomorphism) for the left action of $(\mathfrak{g}, K)$ via right translation. Dually, we can look at the space of compactly supported functions $C_c^\infty(G)$ with right $(\mathfrak{g}, K)$-action and take $m(W)$ to be (the largest separated quotient of) $C_c^\infty(G) \otimes_{(\mathfrak{g},K)} W$.

In a few weeks, we'll replace $(\mathfrak{g}, K)$-modules by quasi-coherent sheaves on $*/G^{\mathrm{la}}$, which again admit a functor from a category whose objects are more literally $G$-representations and which can be understood more geometrically; we can then interpret these minimal and maximal globalizations as left and right adjoints of this functor.

The $\mathfrak{g}$-action is equivalent to the action of the universal enveloping algebra $U(\mathfrak{g})$. In particular for "nice" representations the center $Z(U(\mathfrak{g}))$ should act simply; we say that a $G$-representation $V$ is quasi-simple if $Z(U(\mathfrak{g}))$ acts on $V^\infty$ by scalars, which occurs in particular if we have a version of Schur's lemma (as above!).

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with Weyl group $W$. By Chevalley's theorem $\mathfrak{h}^*/\!\!/W \simeq \mathfrak{g}^*/\!\!/G$, so in particular both sides are independent of the choice of $W$.

By the Harish-Chandra isomorphism we can identify $Z(U(\mathfrak{g}))$ with $\mathrm{Sym}(\mathfrak{h})^W \simeq \mathrm{Sym}(\mathfrak{g})^G$, so the central character given by the action on $V^\infty$ is equivalent to a class $\xi$ in $\mathfrak{h}^*/\!\!/W \simeq \mathfrak{g}^*/\!\!/G$; this is the infinitesimal character of $V$. Explicitly, for a Borel $\mathfrak{b}$ containing $\mathfrak{h}$ with sum of roots $2\rho$, $\lambda \in \mathfrak{h}^*$, and representation $V$ with highest weight $\lambda$, the infinitesimal character of $V$ is (the $W$-orbit of) $\lambda + \rho$.

Let's focus on the example of $\mathrm{GL}_2(\mathbb{R})$. In fact it is often easier to work with $\mathrm{SL}_2(\mathbb{R})$ and then add a parameter (a familiar approach from the theory of automorphic forms for $\mathrm{GL}_2$ vs. $\mathrm{SL}_2$).

If $G = \mathrm{SL}_2(\mathbb{R})$, its maximal compact subgroup is $K = \mathrm{SO}_2(\mathbb{R}) \simeq S^1$. We can choose a basis $\{e, h, f\}$ for $\mathfrak{sl}_2$ with $\mathrm{Lie}(K)_{\mathbb{C}}$ spanned by $h$ and standard relations $[h, e] = 2e$, $[f, h] = 2f$, and $[e, f] = h$. Setting $\Delta = \frac{1}{2}h^2 + fe + ef$ gives $Z(U(\mathfrak{sl}_2)) \simeq \mathbb{C}[\Delta]$. For $Z(U(\mathfrak{gl}_2))$, we (freely) adjoin an extra variable.

Fix a line $L \subset \mathbb{R}^2$, determining a Borel subgroup $B \subset G = \mathrm{SL}_2(\mathbb{R})$ and inducing a decomposition $\mathbb{R}^2 \simeq L \oplus L'$. We have a decomposition $B = MAN$ where $M \simeq \{\pm 1\}$, $A = \mathbb{R}^\times_{>0}$, and $N = \mathrm{Aut}_G(L')$. The relevant characters are then $(\epsilon, \lambda) : M \times A \to \mathbb{C}^\times$ sending $(m, a) \mapsto m^\epsilon a^\lambda$ for $\epsilon \in \{0, 1\}$ and $\lambda \in \mathbb{C}^\times$. The parabolic induction $\mathrm{Ind}_B^G(\epsilon, \lambda)$ can then be understood as the space of continuous functions $f : \mathbb{R}^2 - \{0\} \to \mathbb{C}$ such that $f(av) = |a|^{-\lambda-1}\mathrm{sign}(a)^\epsilon f(v)$ for any $a \in \mathbb{R}^\times$ and nonzero $v \in \mathbb{R}^2$. We can calculate that $Z(U(\mathfrak{sl}_2))$ acts by $\Delta \mapsto \frac{\lambda^2-1}{2}$.

If we further ask that $K \simeq S^1$ acts by weight $n$, i.e. for $k = k_\theta$ and $a > 0$ we have $f(akv) = a^{-\lambda-1}e^{-in\theta}f(v)$, this cuts out a 1-dimensional subspace of $\mathrm{Ind}_B^G(\epsilon, \lambda)$, spanned by some function we call $f_n$ (up to scalars). If $V(\epsilon, \lambda) = \mathrm{Ind}_B^G(\epsilon, \lambda)^{\infty,(K)}$, then one can show that $V(\epsilon, \lambda)$ is the span of $\{f_n\}_{n \equiv \epsilon \pmod 2}$.

To modify things for $\mathrm{GL}_2(\mathbb{R})$, we get two factors of each of $\{\pm 1\}$ and $\mathbb{R}^\times_{>0}$ and proceed similarly with $\epsilon_1, \epsilon_2, \lambda_1, \lambda_2$.

We next want to classify the irreducible $(\mathfrak{g}, K)$-modules. For any $(\mathfrak{g}, K)$-module $V$ the $K$-action gives a weight decomposition

$$V = \bigoplus_n V(n).$$

The elements $e, f \in \mathfrak{g}$ change the weights: $e : V(n) \to V(n+2)$ and $f : V(n) \to V(n-2)$ such that $[e, f] = h$ acts by $n$ (i.e. the derivative of the $K$-action of weight $n$).

If $V$ is irreducible with infinitesimal character $\Delta \mapsto \xi$, we say its parity is given by the sign of the action of $-1 \in K$; if $V$ is odd then $V(n)$ vanishes for $n$ even and vice versa. The parity

$\epsilon = \epsilon(V)$ together with the infinitesimal character $\xi$ classify the irreducible $(\mathfrak{g}, K)$-modules, up to some subtleties:

- If $\xi = \frac{1}{2}\ell(\ell+1)$ for some integer $\ell \geq 0$ congruent to $\epsilon$ modulo 2, then there are three possibilities: $V \simeq \mathrm{Sym}^\ell \mathrm{Std}_\mathbb{C}$, $V \simeq D_\ell^+ = V_{>\ell} = \bigoplus_{\substack{n>\ell \\ n\equiv\epsilon \pmod 2}} V(n)$, or $V \simeq D_\ell^- = V_{<\ell}$. We call $D_\ell^+$ and $D_\ell^-$ the holomorphic and antiholomorphic discrete series respectively, corresponding to (anti)holomorphic modular forms of weight $\ell+2$. These fit into short exact sequences
$$0 \to D_\ell^+ \oplus D_\ell^- \to V(\epsilon, \ell+1) \to \mathrm{Sym}^\ell \mathrm{Std} \to 0,$$
$$0 \to \mathrm{Sym}^\ell \mathrm{Std} \to V(\epsilon, -\ell-1) \to D_\ell^+ \oplus D_\ell^- \to 0.$$
We can realize them on the two connected components of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ as global sections of $\mathcal{O}(-\ell-2) \simeq \omega_{\mathbb{P}^1}(-\ell)$.
- If $\xi = -\frac{1}{2}$ and $\epsilon = 1$ we have either $V \simeq D_0^+ = V_{\geq 1}$ or $V \simeq D_0^- = V_{\leq -1}$, the limits of discrete series representations. In this case we have a decomposition
$$D_0^+ \oplus D_0^- \simeq V(1, 0).$$
- If neither of the above conditions hold, there is a unique $(\mathfrak{g}, K)$-module with parity $\epsilon$ and infinitesimal character $\xi$, given by $V(\epsilon, \lambda)$ for any $\lambda$ with $(\lambda+1)^2 = \xi$.

In the case $G = \mathrm{GL}_2(\mathbb{R})$, we have a similar classification with an extra parameter in the central character and some modifications to which of the above are irreducible.

In particular, in both cases: everything is either a principle series $V(\epsilon, \lambda)$, a discrete series $D_\ell^\pm$, a limit of a discrete series $D_0^\pm$, or finite-dimensional. This is notably simpler than the $p$-adic case, where we also have "special" and supercuspidal representations. (Strictly speaking we should put some conditions on $\lambda$, namely either $\lambda \in i\mathbb{R}_{>0}$ or $\lambda \in (0, 1)$.)

There is a more general way of understanding this sort of classification via Beilinson–Bernstein localization, which we touch on briefly and will return to in a few weeks. Fix a character $\xi$ corresponding to $\lambda + \rho \in \mathfrak{h}^* /\!\!/ W \simeq \mathfrak{g}^* /\!\!/ G$ as before, and let $(\mathfrak{g}, K)\text{-}\mathbf{Mod}_\xi$ be the category of $(\mathfrak{g}, K)$-modules with infinitesimal character $\xi$. There is a localization functor
$$(\mathfrak{g}, K)\text{-}\mathbf{Mod}_\xi \to D_\lambda\text{-}\mathbf{Mod}(\mathrm{Fl}_G)^{K_\mathbb{C}}$$
to $K_\mathbb{C}$-invariant $\lambda$-twisted $D$-modules for a suitable flag variety $\mathrm{Fl}_G$. For the case $\mathrm{SL}_2$, the flag variety is $\mathbb{P}^1_\mathbb{C}$, which has three $K_\mathbb{C}$-orbits given by two fixed points $\{0\}$, $\{\infty\}$ and their complement $U = \mathbb{P}^1 - \{0, \infty\}$. For each fixed $\ell$, the two points correspond to the discrete series representations; the complement $U$ corresponds to $\mathrm{Sym}^\ell \mathrm{Std}$ when $\ell \equiv \epsilon \pmod 2$ and to the principal series representation otherwise.

## 2. Real local Langlands

Our goal is to have some parametrization of admissible $G$-representations, modulo infinitesimal equivalence, in terms of L-parameters, which should be certain representations of a Weil group closely related to the absolute Galois group of our archimedean local field. The first thing to do is to say what this Weil group is.

We only have two possible archimedean local fields, $\mathbb{R}$ and $\mathbb{C}$, and so we can define the Weil groups by hand: $W_\mathbb{C} = \mathbb{C}^\times$ and $W_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ with the relations $j^2 = -1$, $jzj^{-1} = \overline{z}$. For either $F = \mathbb{R}$ or $F = \mathbb{C}$ the Weil groups $W_F$ fit into short exact sequences

$$0 \to \overline{F}^\times \to W_F \to \mathrm{Gal}(\overline{F}/F) \to 0.$$

These are notably more interesting than the unmodified absolute Galois groups, which are rather trivial for archimedean fields. Moreover they fit well into the expectations of local class field theory: in both cases $W_F^{\mathrm{ab}} \simeq F^\times$ (trivial for $F = \mathbb{C}$, and for $F = \mathbb{R}$ we can compute that the commutator subgroup is $\{z_1 j z_2 z_1^{-1} z_2^{-1} j^{-1}\} = \{z_1 j z_1^{-1} j^{-1}\} = \{z/\overline{z}\} = S^1$ and so $W_\mathbb{R}^{\mathrm{ab}} \simeq (\mathbb{C}^\times \rtimes \mathbb{Z}/2)/S^1 \simeq \mathbb{R}^\times_{>0} \times \mathbb{Z}/2 \simeq \mathbb{R}^\times$).

This is the special case of $G = \check{G} = \mathrm{GL}(1)$ of the following more general correspondence. A Langlands parameter for $F \in \{\mathbb{R}, \mathbb{C}\}$ is a continuous homomorphism $\varphi : W_F \to {}^L G = \check{G} \rtimes \mathrm{Gal}(\overline{F}/F)$, which in the real case we require to take $j$ to the nontrivial connected component and $\mathbb{C}^\times$ to semisimple elements in the trivial connected component. In general, we should also impose a "relevance" condition, but for quasi-split groups this is trivial and we'll ignore it. Let $\Phi(G)$ be the set of Langlands parameters modulo $\check{G}$-conjugacy, and observe that for GL(1) these are just characters of $W_F$.

Write $\Pi(G)$ for the set of irreducible admissible $(\mathfrak{g}, K)$-modules. Then there is a local Langlands map $\Pi(G) \to \Phi(G)$ with various properties:

- it should have finite nonempty fibers;
- for GL(1) it should recover local class field theory;
- it should be compatible in suitable senses with products and parabolic induction;
- it should be compatible with infinitesimal characters in the following sense: if $V$ is a $(\mathfrak{g}, K)$-module with infinitesimal character $\xi$, the restriction of the Langlands parameter of $V$ to $\mathbb{C}^\times$ should be of the form $z \mapsto z^\xi \overline{z}^\mu$ for some $\mu \equiv \xi \in X_*(T)_\mathbb{C}/X_*(T)$.

This is far from exhaustive, but gives enough conditions to check to informally have a good sense of whether a "naturally constructed" map is likely to be the "right" one.

We'll ultimately want to recover $\Phi(G)$ as functions on a moduli space of Langlands parameters $Z^1_{\check{G}}/\check{G}$ and $\Pi(G)$ as certain sheaves on a classifying stack for $G$.

Very very vaguely, the construction is via parabolic induction from the discrete case: a Langlands parameter $\varphi$ is discrete if its image is not contained in (the Langlands dual of) any proper parabolic. Its restriction to $\mathbb{C}^\times \subset W_\mathbb{R}$ is a cocharacter of $\check{G}$ and thus a character $\xi$ of $G$; the fiber of the local Langlands correspondence over $\varphi$ is given by the discrete series with infinitesimal character $\xi$.

If $\varphi$ is not discrete, then it factors through some parabolic ${}^L P$ in ${}^L G$; take the smallest such parabolic. By semisimplicity, the image of $\varphi$ actually lands in the Levi ${}^L M$, and by minimality viewing $\varphi$ as a Langlands parameter for $M$ it is discrete, and so we can attach to it an L-packet of discrete series for $M$. We then take the parabolic induction of these discrete series to $G$. These need not be irreducible, but as above should have a unique irreducible quotient, the Langlands quotient; these roughly form the L-packet over $\varphi$.

The main result of the seminar that we'll ultimately want to prove is "nonabelian real Lubin–Tate theory" for GL(2), describing how we can think of the combination of the local Langlands correspondence and Jacquet–Langlands as the action of a Hecke operator on a suitable stack $\mathrm{Bun}_{\mathrm{GL}_2}$. Thus we also want to say something about Jacquet–Langlands in the classical case. This is a bijection between irreducible discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ (i.e. unitary subrepresentations of $L^2(\mathrm{GL}_2(\mathbb{R}))$) and irreducible smooth representations of $\mathbb{H}^\times$, preserving central characters (and satisfying various other compatibilities, e.g. on trace distributions). After complexification $\mathbb{H}^\times \simeq \mathrm{GL}_2$ and so it has a standard representation on $\mathbb{C}^2$, as well as its symmetric powers; meanwhile discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ with central character $\omega = \mathrm{sign}^\epsilon \omega^+$ for $\omega^+$ the restriction to $\mathbb{R}_{>0}$ and $\epsilon \in \{0, 1\}$ are of the form $\omega^+ \boxtimes (D_n^+ \oplus D_n^-)$ for $n \geq 2$ congruent to $\epsilon$ modulo 2 and $D_n^\pm$ the holomorphic/antiholomorphic discrete series representation of weights $\pm n, \pm(n+2), \ldots$. The Jacquet–Langlands correspondence then takes $\omega^+ \boxtimes (D_n^+ \oplus D_n^-)$ to $\omega^+ \boxtimes \mathrm{Sym}^{n-2}(\mathbb{C}^2)$.

Ultimately, we'll want to think of $\mathrm{GL}_2(\mathbb{R})$-representations as roughly sheaves on $*/\,\mathrm{GL}_2(\mathbb{R})$ and similarly for $\mathbb{H}^\times$-representations; so we might hope that this correspondence arises by pull-push from some correspondence between the two classifying stacks. Following the $p$-adic version and noting that $\mathbb{H}^\times$ is an inner form of $\mathrm{GL}_2(\mathbb{R})$, we expect that there should be a Hecke stack classifying a $\mathrm{GL}_2(\mathbb{R})$-torsor $\mathcal{E}$, an $\mathbb{H}^\times$-torsor $\mathcal{E}'$, and in some sense a modification between them of prescribed type. In fact this modification should occur at some degree 1 divisor on a mysterious archimedean analogue of the Fargues–Fontaine curve, and so we imagine that we should actually get a correspondence

$$\begin{tikzcd} & \mathrm{Hk}_\mu \arrow[dl] \arrow[dr] & \\ */\,\mathrm{GL}_2(\mathbb{R}) & & */\mathbb{H}^\times \times \mathrm{Div}^1 \end{tikzcd}.$$

Thus for a representation of $\mathrm{GL}_2(\mathbb{R})$ we should get not only an $\mathbb{H}^\times$-representation (corresponding to the Jacquet–Langlands image) but also a sheaf on $\mathrm{Div}^1$, which we will see is the right analogue of a Langlands parameter and is given by the local Langlands correspondence. Making this precise and giving a purely local proof is one of the main goals of the seminar.