Locally analytic representations of real groups

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1. Definitions and first properties

Let G be a real Lie group. Viewed as a real-analytic manifold, this gives rise to an analytic stack G^{la} over $\mathbb{C}_{\mathrm{gas}}$, which in fact is a group object in analytic stacks.

The goal of today's talk is to study the category of locally analytic G-representations $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$, where as usual in this seminar $* = \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$. Much of this will have to do with the geometry of the corresponding classifying stack $*/G^{\mathrm{la}}$. Our first result is that it is cohomologically smooth:

Proposition 1. *Let $(1 \subset G^{\mathrm{la}})^\dagger$ be the overconvergent neighborhood of the identity, giving a subgroup in analytic stacks. Both maps in the correspondence $*/G^{\mathrm{la}} \leftarrow */(1 \subset G^{\mathrm{la}})^\dagger \to *$ are cohomologically smooth of dimension* $\dim G$*. In particular, the structure map $*/G^{\mathrm{la}} \to *$ is cohomologically smooth of dimension* 0*, and its dualizing complex is the modulus character of G concentrated in degree* 0.

Here the modulus character of G is the absolute value of the determinant of the adjoint action of G on its Lie algebra.

Proof. Since $(1 \subset G^{\mathrm{la}})^\dagger = G^{\mathrm{la}} \times_{G_{\mathrm{Betti}}} *$, $*/(1 \subset G^{\mathrm{la}})^\dagger \times_{*/G^{\mathrm{la}}} * \simeq G_{\mathrm{Betti}}$, so the pullback of $*/(1 \subset G^{\mathrm{la}})^\dagger \to */G^{\mathrm{la}}$ is $G_{\mathrm{Betti}} \to *$ which is cohomologically smooth since G is smooth.

For the smoothness of $*/(1 \subset G^{\mathrm{la}})^\dagger \to *$, let $\widetilde{G}$ be a small smooth complex-analytic neighborhood of G, with analytic de Rham stack $\widetilde{G}/(1 \subset G^{\mathrm{la}})^\dagger \simeq \widetilde{G}^{\mathrm{an}}_{\mathrm{dR}}$. Since $\widetilde{G}$ is smooth, the projection $\widetilde{G}/(1 \subset G^{\mathrm{la}})^\dagger \to */(1 \subset G^{\mathrm{la}})^\dagger$ is smooth; on the other hand by analytic Riemann–Hilbert $\widetilde{G}/(1 \subset G^{\mathrm{la}})^\dagger \simeq \widetilde{G}^{\mathrm{an}}_{\mathrm{dR}} \simeq \widetilde{G}_{\mathrm{Betti}} \to *$ is smooth, so we conclude. Comparing dimensions shows that the dualizing sheaf of $*/G^{\mathrm{la}} \to *$ is invertible and concentrated in degree 0, hence it is cohomologically smooth of relative dimension 0.

We now try to sketch how to identify the dualizing complex with the modulus character. A priori, it is some character of G. Consider first the case where $G \simeq \mathbb{R}^n$ is abelian; by Künneth the computation reduces to the case $n = 1$. As above, we have the correspondence $*/\mathbb{R}^{\mathrm{la}} \leftarrow */\mathbb{G}_a^\dagger \to *$, where we recall that $\mathbb{G}_a^\dagger = (0 \subset \mathbb{G}_a^{\mathrm{an}})^\dagger$, and $*/\mathbb{G}_a^\dagger \to */\mathbb{R}^{\mathrm{la}}$ base changes to $\mathbb{R}_{\mathrm{Betti}} \to *$ as above, so it suffices to find the dualizing complexes for this and for $*/\mathbb{G}_a^\dagger \to *$, or equivalently the actions of $\mathbb{R}^\times$ on the dualizing complexes. We find that $\mathbb{R}^\times$ acts by the sign character on $\mathbb{R}_{\mathrm{Betti}}$ and by multiplication on $*/\mathbb{G}_a^\dagger$, and therefore by their product on $*/\mathbb{R}^{\mathrm{la}}$, i.e. by the modulus character. More generally on $(\mathbb{R}^n)^{\mathrm{la}}$ the action of $\mathrm{SL}_n(\mathbb{R})$ must be trivial and so the character factors through $\mathbb{R}^\times$ which is as above, so the claim holds for $\mathrm{GL}_n(\mathbb{R})$. Finally by degenerating a Lie group to its Lie algebra we find that G acts through $G \to \mathrm{GL}(\mathrm{Lie}(G))$, hence by the modulus character as above. □

It will often be useful to us (as classically) to have a fixed maximal compact subgroup $K \subset G$, with associated analytic stack $(K \subset G^{\mathrm{la}})^\dagger = G^{\mathrm{la}} \times_{G_{\mathrm{Betti}}} K_{\mathrm{Betti}}$.

We have already seen the utility of studying G^{la} via the overconvergent neighborhood of the identity. In fact we can more generally study the sequence of maps

$$(1 \subset G^{\mathrm{la}})^\wedge \to (1 \subset G^{\mathrm{la}})^\dagger \to (K \subset G^{\mathrm{la}})^\dagger \to G^{\mathrm{la}}, \qquad (*)$$

which we often think of in terms of the corresponding sequence of classifying stacks. These in turn will give rise to a decreasing sequence of categories consisting of all representations of $\mathfrak{g} = \mathrm{Lie}(G)$; "locally analytic" representations of $\mathfrak{g}$; "locally analytic" $(\mathfrak{g}, K)$-modules, where the $\mathrm{Lie}(K)$-representation integrates to a K-representation; and representations which integrate to full G-representations. The rest of this section focuses on the geometry of the group stacks in this sequence and their classifying stacks.

We enumerate some results, with at most sketches of proofs.

Proposition 2. *Let $a : * \to */(1 \subset G^{\mathrm{la}})^\wedge$ be the quotient map. It is cohomologically smooth and surjective, inducing an associative algebra structure on $A := a^! a_! 1$ giving an equivalence*

$$a^! : D(*/(1 \subset G^{\mathrm{la}})^\wedge) \xrightarrow{\sim} D(A_{\mathrm{gas}}),$$

and A is naturally isomorphic to $U(\mathfrak{g})$. In particular since a^* is a twist of $a^!$ it is t-exact for some unique t-structure on $D(*/(1 \subset G^{\mathrm{la}})^\wedge)$.

Proof sketch. The cohomological smoothness follows from noting that the formal completion is a formal ball inside projective space. The first claim about A follows formally from Barr–Beck, and we see (by base change) that A is the compactly supported cohomology of the dualizing complex of $(1 \subset G^{\mathrm{la}})^\wedge$, i.e. the algebra of formal distributions at $1 \subset G$. Sending a function on G defined on a neighborhood of 1 to its X-derivative for $X \in \mathfrak{g}$ and evaluating at 1 gives a map $\mathfrak{g} \to A$ inducing $U(\mathfrak{g}) \to A$, which one can check is an isomorphism. □

This helps us understand the first piece of $(*)$. We next move on to studying the first map:

Proposition 3. *Let $b : */(1 \subset G^{\mathrm{la}}) \to (1 \subset G^{\mathrm{la}})^\dagger$ be the first map in $(*)$. Then $\mathcal{O} = \mathcal{O}_{*/(1 \subset G^{\mathrm{la}})}$ is b-proper with invertible dual, and b^* is fully faithful. The composite*

$$D(*/(1 \subset G^{\mathrm{la}})^\dagger) \xrightarrow{b^*} D(*/(1 \subset G^{\mathrm{la}})^\wedge) \simeq D(U(\mathfrak{g})_{\mathrm{gas}})$$

identifies the source with a full subcategory of the target killed by some idempotent $U(\mathfrak{g})$-algebra. Containment in the image can be checked on one-parameter subgroups, i.e. if $X_1, \ldots, X_d$ form a basis for $\mathfrak{g}$ with corresponding germs of the 1-parameter subgroups $(\mathbb{G}_a^\wedge)_i$, $(\mathbb{G}_a^\dagger)_i \subset G^{\mathrm{la}}$ for the two notions of completion, then an object of $D(*/(1 \subset G^{\mathrm{la}})^\wedge)$ lies in the image of b^* if and only if its restriction to $D(*/(\mathbb{G}_a^\wedge)_i)$ lies in the image of $D(*/(\mathbb{G}_a^\dagger)_i)$.

Note that the map $D(*/(\mathbb{G}_{\mathrm{a}}^\dagger)_i) \to D(*/(\mathbb{G}_{\mathrm{a}}^\wedge)_i)$ is fully faithful from last time.

Proof sketch. The first part is similar to last time, where we did the special case of $G = \mathbb{G}_{\mathrm{a}}$: since $b \circ a : * \to */(1 \subset G^{\mathrm{la}})^\dagger$ is the quotient map, $\mathbb{C}$ is $b \circ a$-proper and so $a_!\mathbb{C}$ is b-proper; and $a_!\mathbb{C}$ is the regular $U(\mathfrak{g})$-representation and we can view the trivial representation $\mathcal{O}$ as a perfect complex with entries $a_!\mathbb{C}$, so it is also b-proper. Therefore b_* satisfies the projection formula and so the full faithfulness of b^* reduces to the identification $b_*\mathcal{O} \simeq \mathcal{O}$, which is similar to the case $G = \mathbb{G}_{\mathrm{a}}$.

The other parts of the proposition are both more technical to prove and less important for us, so we omit the proof; we do mention that the last part reduces to an isomorphism

$$(1 \subset G^{\mathrm{la}})^\dagger \cong \prod_{i=1}^{d} (\mathbb{G}_{\mathrm{a}}^\dagger)_i,$$

which is independently interesting. We also skip a technical lemma about gaseous vector spaces, which we may have cause to regret later. □

We can now study the second term more directly:

Proposition 4. *Let $c = b \circ a : * \to (1 \subset G^{\mathrm{la}})^\dagger$ be the quotient map. It is proper and surjective, and $c^* c_* 1 = \mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$ has a natural coalgebra structure making $D(*/(1 \subset G^{\mathrm{la}})^\dagger)$ naturally isomorphic to the category of comodules over $\mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$. Letting $\mathcal{D}(1 \subset G) = (\mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger)^*$ be its dual, we get a functor to modules over this algebra*

$$D(*/(1 \subset G^{\mathrm{la}})^\dagger) \to D(\mathcal{D}(1 \subset G)_{\mathrm{gas}})$$

which is fully faithful and identifies the image with a full subcategory killed by some idempotent $\mathcal{D}(1 \subset G)_{\mathrm{gas}}$-algebra.

Proof sketch. The first part again follows formally from Barr–Beck. From Proposition 3, we have an inclusion

$$D(*/(1 \subset G^{\mathrm{la}})^\dagger) \to D(*/(1 \subset G^{\mathrm{la}})^\wedge) \simeq D(U(\mathfrak{g})_{\mathrm{gas}}),$$

with right adjoint (isomorphic to) b_*, so it suffices to see that after applying b_* the algebra $\mathcal{D}(1 \subset G)_{\mathrm{gas}}$ over $U(\mathfrak{g})_{\mathrm{gas}}$ identifies with the pullback of $U(\mathfrak{g})$ to $*/(1 \subset G^{\mathrm{la}})^\dagger$, as then b^* factors through a functor as claimed.

We have $c^* c_* 1 = \mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$, which since c is proper has dual $c^! c_* 1 = a^! b^! c_* 1$, so $a^! : D(*/(1 \subset G^{\mathrm{la}})^\wedge) \simeq D(U(\mathfrak{g})_{\mathrm{gas}})$ identifies $\mathcal{D}(1 \subset G)$ with $b^! c_* 1$. On the other hand $U(\mathfrak{g}) \simeq a_! 1$ as above, so we want to identify $b_* a_! 1$ with $b_* b^! c_* 1$, or equivalently (by b-properness of the unit) $b_! a_! 1 = c_! 1$ with $b_! b^! c_* 1$. Since $b_!$ is a twist of b_*, it has a left adjoint $b^\flat$ which is a twist of b^* and so fully faithful, so $b_! b^\flat \simeq \mathrm{id} \simeq b_! b^!$ and so the claim follows from the properness of c. □

We move on to the second map of (∗):

Proposition 5. *The map $d : */(1 \subset G^{\mathrm{la}})^\dagger \to */(K \subset G^{\mathrm{la}})^\dagger$ is a pullback of the quotient $* \to */K_{\mathrm{Betti}}$ and in particular is proper and cohomologically smooth. The category $D(*/(K \subset G^{\mathrm{la}})^\dagger)$ is equivalent to the (∞-)category of comodules over $\mathcal{O}(K \subset G^{\mathrm{la}})^\dagger$. Letting $\mathcal{D}(K \subset G) = (\mathcal{O}(K \subset G^{\mathrm{la}})^\dagger)^*$ be its dual algebra, there is a fully faithful functor*

$$D(*/(K \subset G^{\mathrm{la}})^\dagger) \hookrightarrow D(\mathcal{D}(K \subset G)_{\mathrm{gas}})$$

with image the full subcategory killed by an idempotent algebra, and containment in the image is equivalent to containment in the image of $D(*/(1 \subset G^{\mathrm{la}})^\dagger) \hookrightarrow D(\mathcal{D}(1 \subset G)_{\mathrm{gas}})$ after restriction.

There is a unique t-structure on $D(*/(K \subset G^{\mathrm{la}})^\dagger)$ making d^* t-exact. If K is connected then d^* is fully faithful on the heart, with image the objects of $D(*/(1 \subset G^{\mathrm{la}})^\dagger)^\heartsuit$ on which the representation of $\mathrm{Lie}(K)$ integrates to a locally analytic representation of K, i.e. the objects which after restriction to $D(*/(1 \subset K^{\mathrm{la}})^\dagger)^\heartsuit$ lie in the image of $D(*/K^{\mathrm{la}})^\heartsuit$.

The first part is similar to what we've seen before, and the second is manipulation of adjoint functors; the statement is more interesting than the proof so we omit the latter.

Finally, we reach the end of (∗) and pass to G-representations:

Proposition 6. *The map $e : */(K \subset G^{\mathrm{la}})^\dagger \to */G^{\mathrm{la}}$ is cohomologically smooth, and e^* is fully faithful with left adjoint $e_\sharp$ satisfying a projection formula. There is a unique t-structure on $D(*/G^{\mathrm{la}})$ for which e^* is t-exact.*

Proof sketch. Pulling back e along $* \to */G^{\mathrm{la}}$ gives $(G/K)_{\mathrm{Betti}} \to *$ which is cohomologically smooth, implying the statement on $e_\sharp$; and the full faithfulness of e^* follows from the fact that G/K is contractible. The statement on the t-structure follows from the stability of the image of e^* under truncations, which follows from the analogous statement on $(G/K)_{\mathrm{Betti}} \to *$, pullback along which is t-exact. □

2. Distribution algebras

A different sort of category that we might think of as "gaseous representations of G" would be simply the derived category of $\mathbb{C}_{\mathrm{gas}}[G]$, the free gaseous vector space on the locally compact Hausdorff space G. We would like to relate our category $D(*/G^{\mathrm{la}})$ to this more naive category. It turns out it is convenient to study an intermediary: the derived category of the algebra of compactly supported distributions on G. That is: for any compact Stein subspace $Z \subset G$, we write $\mathcal{D}(Z \subset G) = (\mathcal{O}(Z \subset G)^\dagger)^*$, dual to overconvergent analytic functions on Z; and we set

$$\mathcal{D}_c(G) = \bigcup_{Z \subset G} \mathcal{D}(Z \subset G).$$

The algebra structure on each $\mathcal{O}(Z \subset G)^\dagger$ gives $\mathcal{D}_c(G)$ a coalgebra structure, while convolution makes it an algebra, making it naturally a Hopf algebra; we can give it a gaseous analytic ring structure, denoted by $\mathcal{D}_c(G)_{\mathrm{gas}}$.

Consider the quotient map $* \to */G^{\mathrm{la}}$, inducing $*$-pullback $D(*/G^{\mathrm{la}}) \to D(*) = D(\mathbb{C}_{\mathrm{gas}})$. We claim that in a suitable sense there is a G-action on the image:

Proposition 7. *The $*$-pullback functor $D(*/G^{\mathrm{la}}) \to D(\mathbb{C}_{\mathrm{gas}})$ factors through a fully faithful functor $D(*/G^{\mathrm{la}}) \to D(\mathcal{D}_c(G)_{\mathrm{gas}})$, whose image can be identified with the full subcategory killed by some idempotent $\mathcal{D}_c(G)$-algebra. Containment in the image is equivalent to $D(*/(1 \subset G)^\dagger) \subset D(\mathcal{D}(1 \subset G)_{\mathrm{gas}})$.*

After pullback to $1 \subset G$, this comes from Proposition 4. Scholze gives a way of recovering $\mathcal{D}_c(G)$ cohomologically, but I don't really understand what it's doing.

Via the map $\mathbb{C}_{\mathrm{gas}}[G] \to \mathcal{D}_c(G)$ (which induces the gaseous structure on the target), we can view modules over $\mathcal{D}_c(G)$ as over $\mathbb{C}_{\mathrm{gas}}[G]$, which together with the fully faithful functor of the above proposition gives a functor $D(*/G^{\mathrm{la}}) \to D(\mathbb{C}_{\mathrm{gas}}[G])$. A natural question, to which the answer appears to be unknown is: is this functor fully faithful? Is $\mathcal{D}_c(G)$ an idempotent $\mathbb{C}_{\mathrm{gas}}[G]$-algebra?

3. Minimal and maximal globalization

Recall from a few weeks ago that the classical picture is generally in terms of $(\mathfrak{g}, K)$-modules, which we loosely think of as living in $D(*/G^{\mathrm{la}})$. Note that of course this cannot be the right definition, as there is no dependence on K; we'll come back later to where $(\mathfrak{g}, K)$-modules actually live in this picture. However for many purposes objects of $D(*/G^{\mathrm{la}})$ function like $(\mathfrak{g},K)$-modules: in particular while not the "literal" G-representations we would see in a category like $D(\mathbb{C}_{\mathrm{gas}}[G])$, they form a reasonable replacement with better formal properties.

In the former section, we saw a particular way of recovering a genuine gaseous G-representation from an object of $D(*/G^{\mathrm{la}})$, by $*$-pullback along the quotient map $f : * \to */G^{\mathrm{la}}$. Via the parallel to $(\mathfrak{g},K)$-modules, this would be a canonical globalization, i.e. a G-representation with a given associated $(\mathfrak{g},K)$-module. We saw before that in the classical situation there are two canonical globalizations, given by the minimal and maximal globalizations, which are the left and right adjoints of the map from G-representations to $(\mathfrak{g},K)$-modules. In this situation, there is indeed another natural way we could get a $\mathbb{C}_{\mathrm{gas}}$-module from an object of $D(*/G^{\mathrm{la}})$: simply take !-pullback instead of $*$-pullback. We claim that f^* and $f^!$ give the right notions of minimal and maximal globalization respectively in this context.

Consider for example the notion of parabolic induction. We recall that this is difficult to get a good version of for either classical G-representations or $(\mathfrak{g},K)$-modules, as one has to choose what sort of function spaces to work with. Here, suppose $G = G^{\mathrm{alg}}(\mathbb{R})$ for some connected reductive group G^{alg} with parabolic P^{alg} and Levi M^{alg}, with associated real groups P and M, giving a diagram

$$\begin{array}{ccccc} & & */P^{\mathrm{la}} & & \\ & \swarrow^{q} & & \searrow^{p} & \\ */M^{\mathrm{la}} & & & & */G^{\mathrm{la}} \end{array},$$

where q is cohomologically smooth and p is proper. Then we get a canonical parabolic induction functor

$$p_! q^* : D(*/M^{\mathrm{la}}) \to D(*/G^{\mathrm{la}})$$

without any choices! (Strictly speaking we could replace $p_!$ by p_* or q^* by $q^!$, but since p is proper $p_! = p_*$ and since q is cohomologically smooth $q^* = q^!$ up to a twist, so this gives essentially the same functor.) To get a sense for what globalizations this corresponds to, let's evaluate it on the trivial representation 1 of M.

By proper base change along the Cartesian diagram

$$\begin{array}{ccc} (G/P)^{\mathrm{la}} & \xrightarrow{p'} & * \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ */P^{\mathrm{la}} & \xrightarrow{p} & */G^{\mathrm{la}} \end{array},$$

$f^* p_! q^* 1 \simeq p'_! f'^* q^* 1 = p'_! 1 = \mathcal{O}((G/P)^{\mathrm{la}})$, locally analytic functions on the flag variety G/P. This is exactly the minimal globalization of the $(\mathfrak{g},K)$-module of the principal series representation $\mathrm{Ind}_P^G(1)$.

Using the properness of p, we have $f^! p_! q^* 1 \simeq f^! p_* 1 \simeq p'_* f'^! 1$. Since the dualizing sheaf of $*/P^{\mathrm{la}}$ is the modulus character concentrated in degree 0, this becomes the space of global sections of the sheaf of locally analytic distributions twisted by the modulus character. Choosing a Haar measure we can trivialize it, which also gives $f^! 1 \simeq 1$. The natural map

$f^!1 \otimes f^* - \to f^! -$ adjoint to the projection formula then induces a map $f^* \to f^!$; here this gives the map from locally analytic functions to (twisted) locally analytic distributions, or hyperfunctions, which give the expected maximal globalization.

These recover the minimal and maximal globalizations as gaseous vector spaces, i.e. via pulling back to $D(\mathbb{C}_{\text{gas}})$, with no inherent G-action. However we saw above that f^* actually factors through $D(\mathcal{D}_c(G)_{\text{gas}})$, with a further functor to $D(\mathbb{C}_{\text{gas}}[G])$; we might ask whether something similar is true for $f^!$. Indeed we have the following:

Proposition 8. *The functor $D(*/G^{\text{la}}) \to D(\mathcal{D}_c(G)_{\text{gas}})$ factoring f^* has a right adjoint $D(\mathcal{D}_c(G)_{\text{gas}})$, which itself has a further right adjoint $D(*/G^{\text{la}}) \to D(\mathcal{D}_c(G)_{\text{gas}})$ whose further composite to $D(\mathbb{C}_{\text{gas}})$ is $f^! \otimes (f^!1)^{-1}$.*

Thus f^* and $f^!$ (up to twist) both factor through $D(\mathcal{D}_c(G)_{\text{gas}})$ and so can both be thought of as providing globalizations as G-representations, which by the adjunctions correspond to the minimal and maximal globalizations.

Proof. By Proposition 7, we have a Verdier quotient $D(\mathcal{D}_c(G)_{\text{gas}}) \to D(*/G^{\text{la}})$ killing an idempotent algebra A_G, which formally admits left and right adjoints. The left adjoint is f^* as above; the difficulty is the right adjoint.

Pulling back along $h : */(1 \subset G^{\text{la}})^\dagger \to */G^{\text{la}}$, we get the Verdier quotient $D(U(\mathfrak{g})_{\text{gas}}) \to D(*/(1 \subset G^{\text{la}})^\dagger)$ killing some idempotent $U(\mathfrak{g})$-algebra $A_{\mathfrak{g}}$, and one can formally show that A_G is the base change of $A_{\mathfrak{g}}$ along $U(\mathfrak{g})_{\text{gas}} \to \mathcal{D}_c(G)_{\text{gas}}$; so it suffices to work before this base change.

The Verdier quotient $D(U(\mathfrak{g})_{\text{gas}}) \simeq D(*/(1 \subset G^{\text{la}})^\wedge) \to D(*/(1 \subset G^{\text{la}})^\dagger)$ can be identified with b_*, which we know differs by a twist from $b_!$ and so has right adjoint a twist of $b^!$. In particular, the underlying vector space of the right adjoint, i.e. the pullback along a, is a twist of $a^* b^! h^*$, which by the cohomological smoothness of a and h agrees up to a twist with $f^!$. To identify the twist, we look at the trivial representation, which be trivial after pullback and so the twist must be $(f^!1)^{-1}$. □

4. Relation to $(\mathfrak{g}, K)$-modules

We assume as above that $G = G^{\text{alg}}(\mathbb{R})$, which implies $K = K^{\text{alg}}(\mathbb{R})$. We claim that $(\mathfrak{g}, K)$-modules correspond to $D(*/(K^{\text{alg}} \subset G^{\text{alg}})^\wedge)$: indeed, this category has compact generators given by inductions of irreducible representations of K^{alg} with morphisms as for $(\mathfrak{g}, K)$-modules, which then produces the same category. The category of (algebraic) $(\mathfrak{g}, K)$-modules embeds into a category of "analytic" $(\mathfrak{g}, K)$-modules by $*$-pullback along $a : */(K^{\text{la}} \subset G^{\text{la}})^\wedge \to */(K^{\text{alg}} \subset G^{\text{alg}})^\wedge$, and we have a further natural map $b : */(K^{\text{la}} \subset G^{\text{la}})^\wedge \to */G^{\text{la}}$ as above; this correspondence relates $(\mathfrak{g}, K)$-modules to locally analytic G-representations. One can show that a is proper, with a^* fully faithful as mentioned, and b^* is fully faithful: b is the composite

$$*/(K^{\text{la}} \subset G^{\text{la}})^\wedge \xrightarrow{c} */(K^{\text{la}} \subset G^{\text{la}})^\dagger \xrightarrow{d} */G^{\text{la}},$$

and we've seen that the unit is c-proper with $c_*\mathcal{O} \simeq \mathcal{O}$, so c^* is fully faithful, and d^* is fully faithful with a left adjoint $d_\sharp$ which is a twist of $d_!$. Thus we can study the images of both $(\mathfrak{g}, K)$-modules and locally analytic G-representations inside $D(*/(K^{\text{la}} \subset G^{\text{la}})^\wedge)$, although the images are in general different.

In particular the correspondence lets us move between the two categories. Let $b'_! = d_\sharp c_*$, which is a twist of $b_!$ and a left inverse of b^* (since $b'_! b^* = d_\sharp c_* c^* d^* \simeq d_\# d^* \simeq \mathrm{id}$, using the full faithfulness of c^* and d^*). We note that $D(*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge)$ is a full subcategory of $D(\mathcal{D}(K \subset G)_{\mathrm{gas}})$ and the map $U(\mathfrak{g}) \to \mathcal{D}(K \subset G)$ sends $Z(U(\mathfrak{g}))$ to the center of the distribution algebra, so everything is linear over $Z(U(\mathfrak{g}))$. We can view $Z(U(\mathfrak{g}))$ just as a ring, giving a space $\operatorname{AnSpec} Z(U(\mathfrak{g}))$, but in fact this is naturally a complex-analytic space which viewed as an analytic stack over $\mathbb{C}_{\mathrm{gas}}$ we write as $(\operatorname{AnSpec} Z(U(\mathfrak{g})))^{\mathrm{an}}$; this maps to $\operatorname{AnSpec} Z(U(\mathfrak{g}))$, and we refer to base change along it as localization to the bounded part of $Z(U(\mathfrak{g}))$.

Proposition 9. *The functor $b'_! a^* : D(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \to D(*/G^{\mathrm{la}})$ becomes a t-exact equivalence after localization to the bounded part of $Z(U(\mathfrak{g}))$, with inverse $a_* b^*$.*

The proof is rather long and involved and so we skip it; we mention that it involves a combination of abstract nonsense, the geometric results above, the Cartan decomposition, and analytic Beilinson–Bernstein localization (which we will see next time).

In particular, we get as a corollary a locally analytic upgrading of the correspondence between G-representations and $(\mathfrak{g}, K)$-modules:

Corollary 10. *Let $V \in D(*/G^{\mathrm{la}})^\heartsuit$ be a locally analytic G-representation with bounded infinitesimal character, viewed as a gaseous $\mathcal{D}_c(G)$-module via the minimal globalization (i.e. $*$-pullback). Consider the associated $(\mathfrak{g}, K)$-module $V^{(K)} = a_* b^* V$.*

(a) The map sending a locally analytic G-subrepresentation with bounded infinitesimal character $W \subset V$ to its $(\mathfrak{g}, K)$-module $W^{(K)}$ gives a bijection between the subrepresentations with bounded infinitesimal character of V and sub-$(\mathfrak{g}, K)$-modules of $V^{(K)}$ with bounded infinitesimal character.

(b) The condensed vector space V is quasiseparated if and only if $V^{(K)}$ is. If they are, the closed G-stable subspaces of V each come from locally analytic subrepresentations of V with bounded infinitesimal character, so there is a bijection between closed G-stable subspaces of V and closed $(\mathfrak{g}, K)$-submodules of $V^{(K)}$. In particular $V^{(K)} \subset V$ is dense.

(c) If $V^{(K)}$ is admissible and $Z(U(\mathfrak{g}))$ acts on it via a finite-dimensional quotient, then V is a quasiseparated dual nuclear Fréchet space, the maximal globalization $V^{\max}$ is concentrated in degree 0 and is a quasiseparated nuclear Fréchet space, and the natural map $V \to V^{\max}$ is injective with dense image and induces an isomorphism on K-finite vectors.

If $Z(U(\mathfrak{g}))$ acts via a finite-dimensional quotient on V, then it does on any subrepresentation as well, so we can drop the condition on the infinitesimal character on the subrepresentations.

Proof sketch. Part (a) follows from the t-exact equivalence of Proposition 9. For the second part, any gaseous $\mathcal{D}_c(G)$-module V admits a maximal quasiseparated quotient $V \twoheadrightarrow \overline{V}$ with kernel V°, and one can show that the image of $D(*/G^{\mathrm{la}})$ is stable under the operations $V \mapsto \overline{V}$ and $V \mapsto V^\circ$, and both operations commute with $a_* b^*$, so V is quasiseparated if and only if $V^{(K)} = a_* b^* V$ is.

If V (or equivalently $V^{(K)}$) is quasiseparated and $W \subset V$ is a closed G-stable subspace, the quotient V/W is quasiseparated. The density of $\mathbb{C}[G] \to \mathcal{D}_c(G)$ means that we can promote the G-action to a $\mathcal{D}_c(G)$-action and so we get a short exact sequence of quasiseparated $\mathcal{D}_c(G)$-modules

$$0 \to W \to V \to V/W \to 0,$$

with middle term coming from the image of $D(*/G^{\mathrm{la}})$ and having bounded infinitesimal character. The same then follows for the other pieces; indeed this is equivalent to being killed by some idempotent algebra, which one can check (using the technical lemma we skipped) holds here. The bijection of (b) then follows from that of (a) since it preserves quasiseparatedness and is exact.

The last part then essentially follows from expanding definitions of the various functors and formal properties of quasiseparated dual nuclear Frechet spaces. $\square$