Hodge-Tate and crystalline comparison

Abstract. We briefly review crystalline cohomology and its relationship to prismatic cohomology, and sketch a proof of the crystalline comparison theorem and of the Hodge-Tate comparison theorem as a corollary.

1. Introduction

Recall that last time for a fixed prism (A, I), which we assume for simplicity to be bounded with $I = (d)$, and a smooth A/I-algebra R we get a map of A/I-dgas

$$\eta_R^* : (\Omega^*_{R/(A/I)}, d_{\mathrm{dR}}) \to (H^*(\overline{\mathbb{\Delta}}_{R/A}), \beta_d)$$

extending the structure map $\eta : R \to H^0(\overline{\mathbb{\Delta}}_{R/A})$, where β_d is the Bockstein differential induced by the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{\Delta}}/(d) \xrightarrow{\cdot d} \mathcal{O}_{\mathbb{\Delta}}/(d^2) \to \mathcal{O}_{\mathbb{\Delta}}/(d) \to 0.$$

Our main goal today is to prove that this is an isomorphism.

Our strategy will be as follows: first, we specialize to the case $I = (d) = (p)$, which from our general sketch of prismatic cohomology should correspond roughly to the crystalline setting. To that end we'll review crystalline cohomology and the crystalline-de Rham comparison, and relate crystalline and prismatic cohomology and prove a crystalline-prismatic comparison theorem. We can then deduce the Hodge-Tate comparison in characteristic p, and conclude in general by base change.

It should be noted that all proofs today will really just be sketches, and we'll make a bunch of simplifying assumptions because we can. For the details and rigor see Bhatt and Scholze's paper [2]. The main source for these notes is [1, Lecture VI], which we will follow quite closely.

2. Crystalline cohomology

We're taking $d = p$, so fix a p-torsion-free ring A and a smooth A/p-algebra R. (Bhatt doesn't say so, but I think A should also be p-complete.) Instead of working with the site-theoretic construction, we'll write down an explicit (cosimplicial) complex that computes crystalline cohomology; this should be reminiscent of the complex we wrote down last time to compute prismatic cohomology.

Let P be a(n ind-)smooth A-algebra equipped with a surjection $P \twoheadrightarrow R$, with kernel J. (Bhatt is unclear, but it seems to me that for simplicity we're assuming that A is a $\mathbb{Z}_{(p)}$-algebra; we can tensor .) Recall that for any ring S and ideal $I \subset S$ a PD-structure on I is a set of maps $\gamma_n : I \to S$ for every $n \geq 0$ satisfying a list of properties that make $\gamma_n(x)$ behave like $\frac{x^n}{n!}$. In this case we can define a PD-structure on J just by $\gamma_n(x) = \frac{x^n}{n!}$ after inverting p; then we define $D_J(P)$ to be the p-completion of the subring of $P[\frac{1}{p}]$ generated by P and the images of the $\gamma_n : J \to P[\frac{1}{p}]$. Thus $D_J(P)$ is again p-torsion-free and a surjection $D_J(P) \to R$ defined on P by the map $P \twoheadrightarrow R$ and on $\gamma_n(x)$ for $x \in J$ by $\gamma_n(x) \mapsto 0$.

This is called the PD-envelope of $P \twoheadrightarrow R$, i.e. it has the following universal property.

Lemma 2.1. *Let $D \twoheadrightarrow R$ be a surjection of A-algebras with kernel I, where p is nilpotent on D, and suppose that I has a PD-structure. Then any A-algebra map $P \to D$ commuting with the maps to R extends uniquely to a map $D_J(P) \to D$ compatible with the PD-structures.*

In particular this construction is functorial on the category of A-algebras with surjections to R.

There is a differential $d_{P,J} : D_J(P) \to D_J(P)\widehat{\otimes}_P \Omega^1_{P/A}$ extending the usual de Rham differential $d_P : P \to \Omega^1_{P/A}$ via $d_{P,J}\gamma_n(x) = \gamma_{n-1}(x)\, dx \in \Omega^1_{P/A}[\frac{1}{p}]$; this is a flat connection which we'll use later.

Example 2.2. Take the simplest case $R = A/p$, and set $P = A[x]$ with the surjection defined by $x \mapsto 0$ and the canonical surjection $A \to A/p$, so $J = (p, x)$. Therefore $D_J(P)$ is the p-adic completion in $A[\frac{1}{p}]$ of $A[\gamma_1(x), \gamma_2(x), \ldots]$ (since $\gamma_1(x) = x$). Since each $\gamma_n(x)$ is the unique element of degree n (up to a scalar) using the grading on $P = A[x]$, we have

$$D_J(P) = \widehat{\bigoplus_{n\geq 0}} A \cdot \gamma_n(x)$$

with differential sending $a\gamma_n(x) \mapsto a\gamma_{n-1}(x)$ for $a \in A$, so it has nontrivial cohomology only in degree 0, where it is given by the constants $A \cdot 1$. This is a version of Poincare's lemma.[1]

Fix some P as above. Since P is an A-algebra, we get a cosimplicial A-algebra

$$P^\bullet = \left(P \rightrightarrows P \otimes_A P \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; P \otimes_A P \otimes_A P \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots \right).$$

If we write $P^n = P^{\otimes_A (n+1)}$, we have a multiplication map $\mu : P^n \to P$, which composes with the surjection $P \to R$ to give a map $P^n \to R$ for each n. Let J^n be the kernel of this map, and note that each P^n is again (ind-)smooth as an A-algebra. Applying our construction above, by functoriality we get another cosimplicial A-algebra

$$C^\bullet_{\mathrm{crys}}(R/A) := D_{J^\bullet}(P^\bullet) = \left(D_{J^0}(P^0) \rightrightarrows D_{J^1}(P^1) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; D_{J^2}(P^2) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots \right).$$

We define crystalline cohomology to be the cohomology of the associated complex

$$R\Gamma_{\mathrm{crys}}(R/A) := \mathrm{Tot}(C^\bullet_{\mathrm{crys}}(R/A)) \in D(A).$$

By general nonsense this is a commutative algebra object in $D(A)$. (The reason this agrees with the usual crystalline cohomology is, I think, that for each choice of P $D_{J^\bullet}(P^\bullet)$ is a universal Čech cover of R over A in the crystalline topos.)

Note: it is possible to choose P functorially in R: for example, we can take P to be the free algebra on the underlying set of A. This makes $D_{J^\bullet}(P^\bullet)$ and therefore the crystalline cohomology functorial in R, and so the Frobenius of R induces an endomorphism of $R\Gamma_{\mathrm{crys}}(R/A)$.

Recall the differential $d_{P,J} : D_J(P) \to D_J(P)\widehat{\otimes}_P \Omega^1_{P/A}$. This extends to a differential on $D_J(P)\widehat{\otimes}_P \Omega^*_{P/A}$ making it into a complex. Likewise the same argument as in the prismatic case last time gives us an A/p-linear Bockstein differential β_p on crystalline cohomology. With these differentials, we have the following.

[1] For Bhatt's claim that if we instead used the J-adic completion of P, it seems like we need to be in characteristic p, which we are not...

Theorem 2.3 (Crystalline-de Rham comparison)**.** *There is a natural quasi-isomorphism* $R\Gamma_{\mathrm{crys}}(R/A) \to D_J(P)\widehat{\otimes}_P \Omega^*_{P/A}$.

Choosing P to be a smooth lift of R so that $J = (p)$, tensoring with A/p gives a comparison theorem modulo p.

Corollary 2.4. *There is a natural quasi-isomorphism* $R\Gamma_{\mathrm{crys}}(R/A) \otimes^L_A A/p \to \Omega^*_{R/(A/p)}$ *in* $D(A/p)$.

(Note: naturality does not follow from Theorem 2.3, since we have to choose a lift; but we can get it by reproving the theorem with A/p-coefficients.)

Let $\phi_A : A/p \to A/p$ be the Frobenius, and define the Frobenius twist $R^{(1)} = R\otimes_{A/p,\phi} A/p$, i.e. the pushout

$$\begin{array}{ccc} A/p & \xrightarrow{\phi_A} & A/p \\ \downarrow & & \downarrow \\ R & \longrightarrow & R^{(1)} \end{array} .$$

By the universal property of the pushout, since the diagram

$$\begin{array}{cccc} A/p & \xrightarrow{\phi_A} & A/p & \\ \downarrow & & \downarrow & \searrow \\ R & \longrightarrow & R^{(1)} & \downarrow \\ & \searrow^{\phi_R} & & R \end{array}$$

commutes, where $\phi_R : R \to R$ is the Frobenius on R, there exists a unique ring homomorphism $\phi : R^1 \to R$ making

$$\begin{array}{cccc} A/p & \xrightarrow{\phi_A} & A/p & \\ \downarrow & & \downarrow & \searrow \\ R & \longrightarrow & R^{(1)} & \xrightarrow{\phi} \\ & \searrow^{\phi_R} & & R \end{array}$$

commute. Therefore R is an $R^{(1)}$ algebra and so each $\Omega^i_{R/(A/p)}$ is an R-algebra, and so so is the strictly graded commutative ring $H^*(\Omega^*_{R/(A/p)})$. The A/p-linear Bockstein differential β_p on mod p crystalline cohomology gives by Corollary 2.4 a differential on $H^*(\Omega^*_{R/(A/p)})$ (which we will denote in the same way), making it an A/p-dga. Therefore by the universal property of the de Rham complex the $R^{(1)}$-algebra structure of $H^*(\Omega^*_{R/(A/p)})$ extends to a map of A/p-dgas

$$\mathrm{Cart}^* : (\Omega^*_{R^{(1)}/(A/p)}, d_{\mathrm{dR}}) \to (H^*(\Omega^*_{R/(A/p)}), \beta_p).$$

Theorem 2.5 (Cartier)**.** *This map* Cart^* *is an isomorphism.*

(Note: using Corollary 2.4 this can also be formulated in terms of crystalline cohomology. The asymmetry of $R^{(1)}$ on one side and R on the other remains, but will go away once we switch to prismatic cohomology.)

3. Crystalline vs prismatic cohomology

Recall from last time that if (A, I) is a prism, R is an A/I-algebra, and F_0 is an A-algebra equipped with a surjection to R with kernel J, we can construct the prismatic envelope (F, IF) of (F_0, J) such that the diagram

$$\begin{array}{ccccc} A & \longrightarrow & F_0 & \longrightarrow & F = F_0\left\{\frac{J}{I}\right\}^{\wedge} \\ \downarrow & & \downarrow & & \downarrow \\ A/I & \longrightarrow & R \simeq F_0/J & \longrightarrow & F/IF \end{array}$$

commutes. This gives an object in $(R/A)_{\Delta}$, and in fact it turns out to be a weakly initial object so that, letting $F_0^n = F_0^{\otimes_A(n+1)}$ and J^n be the kernel of the composition of the multiplication map $F_0^n \to F_0$ with the map $F_0 \to R$, the cosimplicial A-algebra

$$F_0\left\{\frac{J^0}{I}\right\} \rightrightarrows F_0^1\left\{\frac{J^1}{I}\right\} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} F_0^2\left\{\frac{J^2}{I}\right\} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

computes prismatic cohomology $\Delta_{R/A}$.

This is very reminiscent of our complex computing crystalline cohomology $D_{J^\bullet}(P^\bullet)$, with the PD-envelope replaced by the prismatic envelope. (In our case, $I = (p)$.) Fortunately we can relate these two notions.

Lemma 3.1. *Let A be a p-torsion-free δ-ring, and let P be a δ-A-algebra which is p-completely flat over A. (For example, ind-smooth.) Let $x_1, \ldots, x_r \in P$ be such that their images in P/p form a regular sequence, and set $J = (p, x_1, \ldots, x_r) \subset P$. Then*

$$P\left\{\frac{\phi(J)}{p}\right\}^{\wedge} = D_J(P),$$

where ϕ is the Frobenius on P associated to the δ-structure.

Note that if we take P free in our construction from section 2, then each $J^n \subset P^n$ satisfies this condition.

Proof sketch. We'll prove this for $P = \mathbb{Z}_p\{x\}$; the general case follows by a base change argument. Recall that $\mathbb{Z}_p\{x\}$ is the free δ-ring over $\mathbb{Z}_p$ on one variable x, given by $\mathbb{Z}_p[x_0, x_1, x_2, \ldots]$ with $x = x_0$ and $\delta(x_n) = x_{n+1}$. In this case $P/p = \mathbb{F}_p\{x\}$, so the only nontrivial possibility for J is $J = (p, x)$. Since ϕ fixes p, the left-hand side of the desired equality is just $P\{\frac{\phi(x)}{p}\} \simeq \mathbb{Z}_p\{x, y\}/(py - \phi(x))$.

This is defined by the pushout

$$\begin{array}{ccc} \mathbb{Z}_p\{z\} & \longrightarrow & \mathbb{Z}_p\{z,y\}/(py-z) \\ \downarrow{\scriptstyle z\mapsto\phi(x)} & & \downarrow{\scriptstyle y\mapsto\frac{\phi(x)}{p}} \\ \mathbb{Z}_p\{x\} & \longrightarrow & \mathbb{Z}_p\left\{x, \frac{\phi(x)}{p}\right\} \end{array} .$$

The left vertical map is essentially the Frobenius on P, and turns out to be flat; it follows that $P\{\frac{\phi(x)}{p}\} \simeq \mathbb{Z}_p\{x,y\}/(py-\phi(x))$ is flat over $\mathbb{Z}_p\{z,y\}/(py-z)$ and therefore p-torsion-free.

Inverting p everywhere, the top map becomes an isomorphism and therefore so doe the bottom map, so $P[\frac{1}{p}] \simeq P\{\frac{\phi(x)}{p}\}[\frac{1}{p}]$. Therefore the inclusion of $P\{\frac{\phi(x)}{p}\}$ into the right-hand side gives an inclusion into $P[\frac{1}{p}]$, and so we can think of it as a subring of $P[\frac{1}{p}]$. We have $\phi(x) = x^p + p\delta(x)$, so $P\{\frac{\phi(x)}{p}\} = P\{\frac{x^p}{p} + \delta(x)\} = P\{\frac{x^p}{p}\}$ as a subring of $P[\frac{1}{p}]$. Thus $P\{\frac{\phi(x)}{p}\}$ is the smallest δ-subring of $P[\frac{1}{p}]$ containing P and $\frac{x^p}{p}$, which we will call C for convenience.

On the other hand, by a similar argument the right-hand side of the desired inequality is $D := D_J(P) = P[\gamma_1(x), \gamma_2(x), \ldots]$. Thus to show $D \subseteq C$ it suffices to show that $\gamma_n(x) \in C$ for every n. We have $\gamma_p(x) = \frac{x^p}{p!} \in C$ as above since $(p-1)!$ is invertible in $\mathbb{Z}_p$ and $\frac{1}{(p-1)}\frac{x^p}{p} = \gamma_p(x)$, and for any n and any $z \in P[\frac{1}{p}]$ we have

$$\gamma_n(\gamma_p(z)) = \frac{(np)!}{n!(p!)^n}\gamma_{np}(z);$$

we can check that the coefficient will always be invertible in $\mathbb{Z}_p$. Therefore $\gamma_{np}(z)$ is in C if and only if $\gamma_n(\gamma_p(z))$ is. Therefore if we have $z \in C$ and $\gamma_{p^k}(z) \in C$ for all $k \geq 1$, we can prove by induction on n that $\gamma_n(z) \in C$ for all n: we can reduce as above to the case where n is coprime to p, in which case the greatest power of p dividing $n!$ is strictly less than $\sum_{k=1}^{\log_p n} \frac{n}{p^k} \leq \frac{n-1}{p-1}$. Therefore if we write $n = pm + r$ for some $r < p$ we have

$$\frac{x^n}{n!} = u\frac{(x^p)^m}{p^k}x^r$$

for some unit $u \in \mathbb{Z}_p^\times$ and some integer $k < \frac{n-1}{p-1}$. In particular $k \leq m$ whenever $n > p$, and so this is in C since $\frac{x^p}{p}$, p, and x are. If $n < p$ then $n!$ is not divisible by p and so $\gamma_n(x)$ is some unit times a power of x and therefore in C.

Thus it suffices to show that $\gamma_{p^k}(x) \in C$ for all $k \geq 1$. This is by induction, since we know that $\gamma_p(x) \in C$. Since C is a δ-ring, it follows that $\delta(\gamma_{p^k}(x)) \in C$ if $\gamma_{p^k}(x) \in C$. We have

$$\delta\left(\frac{x^{p^k}}{(p^k)!}\right) = \frac{1}{p}\left(\frac{\phi(x)^{p^k}}{(p^k)!} - \frac{(x^{p^k})^p}{((p^k)!)^p}\right) = \frac{1}{p}\left(\frac{(x^p + p\delta(x))^{p^k}}{(p^k)!} - \frac{x^{p^{k+1}}}{((p^k)!)^p}\right).$$

Since $(p^k)!$ has $1 + p + p^2 + \cdots + p^{k-1} = \frac{p^k-1}{p-1}$ factors of p, since

$$\frac{(x^p + p\delta(x))^{p^k}}{p^{1+\frac{p^k-1}{p-1}}} = p^{p^k - \frac{p^k-1}{p-1} - 1}\left(\frac{x^p + p\delta(x)}{p}\right)^{p^k} \in C$$

since $\frac{x^p}{p} \in C$ and $\delta(x) \in C$ and this differs from the first term above by a unit of $\mathbb{Z}_p$, combined with the fact that the left-hand side is in C by assumption it follows that

$$\frac{x^{p^{k+1}}}{p((p^k)!)^p} \in C.$$

Again this differs by a unit of $\mathbb{Z}_p$ from $\gamma_{p^{k+1}}(x)$, so this is also in C, and by induction the claim follows.

It remains only to show that $C \subseteq D$. Since $\gamma_p(x) \in D$ and therefore $\frac{x^p}{p} \in D$, it suffices to show that D is a δ-ring, or equivalently that the Frobenius ϕ of P lifts to an endomorphism of D. Certainly it lifts to $P[\frac{1}{p}]$, so we just need to show that it fixes $D \subset P[\frac{1}{p}]$, i.e. that $\phi(\gamma_n(x)) \in D$ for every n. In particular we need to show that $\phi(\gamma_n(x)) - \gamma_n(x)^p \in pD$: being in D shows that it is an endomorphism, and being in pD shows that it is a lift of Frobenius. In fact similarly to above we have $\gamma_{np}(x) = \gamma_p(\gamma_n(x)) = \frac{\gamma_n(x)^p}{p}$ up to a unit of $\mathbb{Z}_p$, so $\gamma_n(x)^p \in pD$ and it suffices to check that $\phi(\gamma_n(x)) \in pD$ for each n; and expanding in terms of δ this becomes an elementary computation of p-adic valuations. □

This allows us to give a more explicit description of the prismatic envelopes.

Corollary 3.2. *Let $(A, (d))$ be a bounded prism, P be a (p,d)-completely flat δ-A-algebra, and $x_1, \ldots, x_r$ be a (p,d)-completely regular sequence relative to A (which is a technical condition I don't want to get into), with $J = (p, d, x_1, \ldots, x_r)$.*

1) *The (p,d)-adic completion E of $P\{\frac{x_1}{d}, \ldots, \frac{x_r}{d}\}$ in the category of δ-rings is (p,d)-completely flat over A.*
2) *This ring E is equal to the prismatic envelope $P\{\frac{J}{d}\}^\wedge$.*
3) *Prismatic envelopes commute with base change along maps of bounded prisms.*

Proof sketch; omit time depending. Base change is easy from the construction of E (apparently), and by flatness E is p-torsion-free and so there is map $(A, (d)) \to (E, (d))$ which is a map of bounded prisms, and it can be verified that it has the desired universal property. Therefore it suffices to prove flatness.

When $(d) = (p)$, we can prove flatness from Lemma 3.1 and properties of divided power algebras; and the more general result is proven by base change. □

We're now more or less ready to prove the crystalline comparison theorem. Set $d = p$, and rewrite our complex computing prismatic cohomology with the notation of section 2 as

$$C^\bullet_{\Delta}(R/A) := \left(P^0\left\{\frac{J^0}{I}\right\} \longrightarrow P^1\left\{\frac{J^1}{I}\right\} \rightrightarrows P^2\left\{\frac{J^2}{I}\right\} \Rrightarrow \cdots \right),$$

so that as last time we have

$$\Delta_{R/A} \simeq \mathrm{Tot}(\mathbb{C}^\bullet_{\Delta}(R/A)).$$

We can again make this functorial by taking P to be e.g. the free δ-A-algebra on $W(R)$.

Theorem 3.3 (Crystalline comparison)**.** *There is a canonical isomorphism*

$$(\phi_A^* \Delta_{R/A})^\wedge \simeq R\Gamma_{\mathrm{crys}}(R/A)$$

of commutative algebra objects in $D(A)$ compatibly with Frobenius.

Proof sketch. For simplicity we'll take $A = \mathbb{Z}_p$, so that ϕ_A is the identity and so the left-hand side is just $\Delta_{R/\mathbb{Z}_p}$. We'll construct this isomorphism by writing down a homotopy equivalence between the two cosimplicial A-algebras the compute each side, $C^\bullet_\Delta(R/A) = P^\bullet\{\frac{J^\bullet}{p}\}^\wedge$ and $D_{J^\bullet}(P^\bullet)$.

By Čech theory, the structure map $A \to P^\bullet$ is a homotopy equivalence of cosimplicial δ-A-algebras since we've chosen it to be free. Thus for any cosimplicial $P^\bullet$-algebra $Q^\bullet$ (which is also an A-algebra) the induced map $\phi_A^* Q^\bullet \to \phi_{P^\bullet}^* Q^\bullet$ is also a homotopy equivalence. In particular we have a homotopy equivalence

$$\phi_A^* C^\bullet_\Delta(R/A) \to \phi_{P^\bullet}^* C^\bullet_\Delta(R/A).$$

Since we're taking $A = \mathbb{Z}_p$, the left-hand side is just $C^\bullet_\Delta(R/A)$; the right-hand side is

$$\phi_{P^\bullet}^* P^\bullet \left\{\frac{J^\bullet}{p}\right\}^\wedge = P^\bullet \left\{\frac{\phi_{P^\bullet}(J^\bullet)}{p}\right\}^\wedge.$$

But by Lemma 3.1 the right-hand side can be identified with $D_{J^\bullet}(P^\bullet)$, which computes $R\Gamma_{\text{crys}}(R/A)$.

We have not explained the compatibility with Frobenius part of the statement, nor will we. □

4. Proof of Hodge-Tate comparison

We are finally ready to sketch a proof of the Hodge-Tate comparison theorem, as stated last time and at the beginning.

Let $(A, (d))$ be a bounded prism, and R be a formally smooth $A/(d)$-algebra. First, assume $(d) = (p)$, and that the Frobenius ϕ_A on A/p is faithfully flat. Recall our map $\eta_R^* : (\Omega^*_{R/(A/(p))}, d_{\text{dR}}) \to (H^*(\overline{\Delta}_{R/A}), \beta_d)$ of $A/(d)$-dgas. Taking the pullback by ϕ_A on both sides and applying Theorem 3.3 (crystalline comparison) and Corollary 2.4 to the right-hand side makes this into a map

$$(\Omega^*_{R^{(1)}/(A/(d))}, d_{\text{dR}}) \to (H^*(\Omega^*_{R/(A/p)}), \beta_p)$$

since $R^{(1)}$ is the pullback of R by ϕ_A, and this agrees with the Cartier isomorphism since everything in sight is canonical; since it differs from the original map only by isomorphisms, we conclude that $\phi_A^* \eta_R^*$ is also an isomorphism. Since ϕ_A^* is faithfully flat it follows that η_R^* is an isomorphism.

In general when $(d) = (p)$ we need a certain étale localization property for prismatic cohomology which allows us to reduce to the case $R = (A/p)[x_1, \dots, x_n]$; then by base change we can reduce to the case $A = \mathbb{Z}_p$, which certainly has faithfully flat Frobenius.

In general, we work at the level of the derived category: we can choose a map

$$\eta_R : \bigoplus_n \Omega^n_{R/(A/(d))} \to \overline{\Delta}_{R/A}$$

inducing η_R^* on cohomology (since we've assumed that R is formally smooth, so each term is finite projective over R). We want to show that η_R is an isomorphism in $D(R)$; this is invariant under (p,d)-completely flat base change, so in particular we can assume $d = \phi(e)$ for some $e \in A$.

We'll assume that d, or equivalently e, is a non-zero-divisor in A/p. Let $D = A\{\frac{d}{p}\}^\wedge = A\{\frac{\phi(e)}{p}\}^\wedge$. By Lemma **??** this is p-torsion-free and coincides with $D_{(e)}(A)$, the p-adically completed divided power envelope of A with respect to (e). We have a structure morphism $\alpha : A \to D$; since the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & D \\ \downarrow & & \downarrow \\ A/(d) & \longrightarrow & R \end{array}$$

commutes, $\alpha(d)$ is in the kernel of the map $D \to R$, which as above is in pD, so $\alpha(d) = pu$ for some $u \in D$; and by the irreducibility of distinguished elements it follows that u is a unit, so α gives a map $(A,(d)) \to (D,(p))$ of bounded prisms. Modulo d, this map factors as $A/(d) \to A/(p,d) = A/(p,\phi(e)) = A/(p,e^p) \to D/p$. Thus for abstract nonsense reasons (the second map is faithfully flat, the first is well-behaved since we assumed that d is not a zero divisor in A/p) p-complete base change of derived (p,d)-complete complexes along α is a conservative functor, i.e. the only morphisms which are mapped to isomorphisms are themselves isomorphisms. By Corollary 3.2, prismatic envelopes commute with base change along α, and since these compute prismatic cohomology it follows that

$$\overline{\Delta}_{R/A}\widehat{\otimes}^L_A D \simeq \overline{\Delta}_{R\widehat{\otimes}_A D/D},$$

and doing the same thing on the de Rham side gives the base changed map

$$\alpha_*\eta_R : (\Omega^*_{R\widehat{\otimes}_A D/(D/p)}, d_{\mathrm{dR}}) \to (H^*(\overline{\Delta}_{R\widehat{\otimes}_A D/D}), \beta_d).$$

In this case we can apply the above to conclude that $\alpha_*\eta_R$ is an isomorphism; and since base change along α is a conservative functor it follows that η_R is also an isomorphism.

The Hodge-Tate comparison allows us to globalize prismatic cohomology using the sheaf properties of differential forms.

Corollary 4.1. *Let X be any formal scheme over* $\mathrm{Spec}(A/(d))$. *There exists a functorial (p,d)-complete commutative algebra object $\Delta_{X/A} \in D(X,A)$ equipped with a ϕ_A-linear endomorphism ϕ_X such that*

- *for any affine open $U = \mathrm{Spf}(R) \subseteq X$ there is a natural isomorphism between $R\Gamma(U, \Delta_{X/A})$ and $\Delta_{R/A}$ carrying ϕ_X to ϕ_R, and*
- *if we set $\overline{\Delta}_{X/A} = \Delta_{X/A} \otimes^L_A A/(d) \in D(X, A/(d))$, then $\overline{\Delta}_{X/A}$ is a perfect complex on X and we have canonical isomorphisms $\Omega^n_{X/A} \to H^n(\overline{\Delta}_{X/A})$ for each n, compatible with the Hodge-Tate comparison theorem for the local isomorphisms in the previous part.*

References

[1] Bhargav Bhatt. Samuel Eilenberg Lectures at Columbia University, lecture notes: prismatic cohomology, 2018. URL: `http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/lecture6-hodge-tate-and-crystalline-comparison.pdf`.

[2] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *arXiv preprint arXiv:1905.08229*, 2019.