Hilbert modular Shimura varieties and Hecke operators

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1. Introduction

Let's start by recalling some notation: F is a totally real number field with ring of integers $\mathcal{O}$, I is the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ with $|I| = [F : \mathbb{Q}] = n$, $G = \operatorname{Res}_{\mathcal{O}/\mathbb{Z}} \mathrm{GL}(2)$, i.e. $G(R) = \mathrm{GL}_2(R \otimes_{\mathbb{Z}} \mathcal{O})$, and $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}}$ is the Deligne torus.

Let $h_0 : \mathbb{S} \to \mathrm{GL}(2) = G_{/\mathbb{R}}$ be the homomorphism of real algebraic groups defined on real points by $\mathbb{S}(\mathbb{R}) \ni a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. We define X to be the conjugacy class of h_0 under the action of $G(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R})^I$, with a corresponding action of $G(\mathbb{R})$ on X by conjugation. The stabilizer of h_0 for this action is the centralizer of h_0 in $G(\mathbb{R})$, which is just the product of the maximal compact subgroup $K(\mathbb{R})^+$ of the connected component $G(\mathbb{R})^+$ of the identity (isomorphic to $\mathrm{SO}_2(\mathbb{R})^I$) with its center $Z(\mathbb{R}) \simeq (\mathbb{R}^\times)^I$, so the connected component X^+ of h_0 is isomorphic to $G(\mathbb{R})^+/(Z \times K(\mathbb{R})^+) \simeq (\mathrm{GL}_2(\mathbb{R})/(\mathbb{R}^\times \times \mathrm{SO}_2(\mathbb{R})))^I \simeq \mathcal{H}^I$, with the action of $g = (g_\sigma)_\sigma \in G(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R})^I$ on $\mathcal{H}^I$ given by a copy of the fractional linear action for each $\sigma \in I$. Explicitly, we can give this isomorphism by $g h_0 g^{-1} \mapsto g \cdot \mathbf{i}$ where $\mathbf{i} = (i, \ldots, i) \in \mathcal{H}^I$ via this action. Therefore the whole space X is some finite disjoint union of spaces isomorphic to $\mathcal{H}^I$, so for each arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ the quotient $\Gamma \backslash X$ is a finite disjoint union of connected Hilbert modular varieties.

The pair (G, X) is then a Shimura datum, and admits a Shimura variety $\mathrm{Sh}(G, X) = \varprojlim_K \mathrm{Sh}_K(G, X)$ for every compact open subgroup K of $G(\mathbb{A}_f)$ with complex points

$$\mathrm{Sh}(G, X)(\mathbb{C}) = \varprojlim_{K} G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f))/K$$

where $\mathbb{A}_f$ are the finite adeles. There is an obvious action of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)(\mathbb{C})$ by right multiplication. In fact $\mathrm{Sh}(G, X)$ has a unique canonical model defined over $\mathbb{Q}$, which has an interpretation as a moduli space of abelian varieties.

Our goal for today is first to explain this moduli interpretation. We will then look at the Igusa tower again and derive a q-expansion principle from its irreducibility. Finally, we will interpret the Hecke operators as correspondences on the Shimura varieties.

2. Abelian varieties up to isogeny

Let $V = F^2$, and write $V(\mathbb{A}_f) = V \otimes_{\mathbb{Q}} \mathbb{A}_f$, which is a free module of rank 2 over $\mathbb{A}_{F,f} = \mathbb{A}_f \otimes_{\mathbb{Q}} F$.

Consider the category $\mathcal{A}_F^{\mathbb{Q}}$ (fibered over $\mathbf{Sch}_{\mathbb{Q}}$) of abelian schemes with real multiplication by $\mathcal{O}$, with morphisms $\operatorname{Hom}_F^{\mathbb{Q}}(A, A') = \operatorname{Hom}_{\mathcal{O}}(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$, so that isogenies are isomorphisms. For an abelian scheme A over S and a geometric point $s \in S$, we consider the Tate module $\mathcal{T}(A) = \mathcal{T}_s(A) = \varprojlim_N A[N](k(s))$, and define $V(A) = V_s(A) = \mathcal{T}(A) \otimes_{\mathbb{Z}} \mathbb{A}_f$. This latter module $V(A)$ is a free $\mathbb{A}_{F,f}$-module of rank 2, with an $\widehat{\mathcal{O}} = \prod_\ell \mathcal{O}_\ell$-stable lattice $\mathcal{T}(A)$. A full level structure on A is an isomorphism of $\mathbb{A}_{F,f}$-modules $\eta : V(\mathbb{A}_f) \xrightarrow{\sim} V(A)$, after picking a geometric point s in each connected component of S.

For a closed subgroup $K \subset G(\mathbb{A}_f)$, we can define a level K-structure to be a K-orbit $\overline{\eta} = \eta K$, where $g \in G$ acts on η by precomposition.

We can also consider polarizations $A \to A^\vee$. We consider two polarizations λ, λ' to be equivalent if they agree up to some totally positive $a \in F^\times$, i.e. $\lambda = \lambda' \circ a$. Note that we need our category $\mathcal{A}_F^{\mathbb{Q}}$ with the extra morphisms given by tensoring with $\mathbb{Q}$ in order for this to be well-defined for non-integral a. Write $\overline{\lambda}$ for the equivalence class of λ.

For a fixed closed subgroup K, we define the functor $\mathcal{E}_K : \mathbf{Sch}_{\mathbb{Q}} \to \mathbf{Set}$ sending S to the set of isomorphism classes of triples $(A, \overline{\lambda}, \overline{\eta})$ where A is an abelian variety over S, $\overline{\lambda}$ is an equivalence class of polarizations defined over S, and $\overline{\eta}$ is a level K-structure defined over S, satisfying a further condition (that the characteristic polynomial of $\alpha \in \mathcal{O}$ on the sheaf of Lie algebras over A is given by the image of the obvious product in $\mathcal{O}_S[t]$) that we will generally ignore.

It turns out that the canonical model of $\mathrm{Sh}(G, X)$ over $\mathbb{Q}$ represents the functor $\mathcal{E}_1$, i.e. $\mathcal{E}_K$ in the case where K is the trivial group. Since $G(\mathbb{A}_f)$ acts naturally on $V(\mathbb{A}_f)$, it also acts on the level structure by precomposition and thus acts on the right on $\mathrm{Sh}(G, X)$. For a sufficiently small open subgroup K (such that the test objects have no nontrivial automorphisms), we can define $\mathrm{Sh}_K(G, X) = \mathrm{Sh}(G, X)/K$ (for the canonical $\mathbb{Q}$-model), which then represents $\mathcal{E}_K$. Indeed, we can define $\mathrm{Sh}(G, X)$ and $\mathrm{Sh}_K(G, X)$ as representing these functors, up to showing that they are representable, and the description above follows.

The proof of representability proceeds roughly by showing that the moduli problem is equivalent to another which decomposes as the disjoint union of (modifications of) the moduli problem for $\Gamma(N)$, which we know is representable for N sufficiently large over $\mathbb{Q}(\mu_N)$; we then assemble all the conjugates to get something defined over $\mathbb{Q}$.

We could also ask for an integral model of $\mathrm{Sh}(G, X)$ or $\mathrm{Sh}_K(G, X)$. We can find one by altering the moduli problem slightly: fix a set of primes Σ, and define the category $\mathcal{A}_F^{\Sigma}$ to have the same objects as $\mathcal{A}_F^{\mathbb{Q}}$ with morphisms given by tensoring with $\mathbb{Z}_{(\Sigma)}$ instead of $\mathbb{Q}$, i.e. we invert all primes except those in Σ, and define the moduli problem $\mathcal{E}_K^{\Sigma}$ in the same way as $\mathcal{E}_K$ by suitably adding the phrase "prime to Σ." We get corresponding representing schemes $\mathrm{Sh}_K^{\Sigma}(G, X)$ and $\mathrm{Sh}^{\Sigma}(G, X)$, defined over $\mathbb{Z}_{(\Sigma)}$. In particular, we are interested in the case where $\Sigma = \{p\}$, so that $\mathrm{Sh}_K^p(G, X)$ and $\mathrm{Sh}^p(G, X)$ are defined over $\mathbb{Z}_{(p)}$. We can check that test objects over R/p lift to R for $\mathbb{Z}_{(p)}$-algebras R, so it follows that $\mathrm{Sh}^p(K(G, X)$ and $\mathrm{Sh}^p(G, X)$ are smooth over $\mathbb{Z}_{(p)}$.

3. Irreducibility of the Igusa tower

Let $\mathcal{W}$ be the strict Henselization of $\mathbb{Z}_{(p)}$ in $\overline{\mathbb{Q}}$, which has residue field $\overline{\mathbb{F}}_p$. We are interested in the Igusa tower $T_\alpha = T_{1,\alpha/\overline{\mathbb{F}}_p}$ over the toroidal compactification M of the Hilbert-Blumenthal moduli $\mathcal{M} = \mathcal{M}(c, \Gamma(N))_{/\overline{\mathbb{F}}_p}$; write $T_\alpha^\circ = T_\alpha \times_M \mathcal{M}$ for its base change to $\mathcal{M}$.

The main result here is that the T_α°, and therefore also T_α, are irreducible, when N is prime to p. The proof however is long and difficult and so I prefer to skip it, apologies to anyone who was hoping to see it (you can go read Hida). Hida's proof applies the computation of the automorphism group of the canonical model of $\mathrm{Sh}(G, X)$, which we also skipped.

Instead, let's prove a corollary, namely a q-expansion principle. Let $\mathfrak{a}, \mathfrak{b} \subset F$ be fractional ideals prime to p with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ for a fixed choice of polarization ideal $\mathfrak{c}$. We have a semi-

AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$, given by the algebraization of the formal quotient $(\widehat{\mathbb{G}_\mathrm{m}} \otimes \mathrm{Hom}(\mathfrak{a}, \mathbb{Z}))/q^{\mathfrak{b}}$, which away from the cusps coincides with the universal abelian scheme over $\mathcal{M}$ restricted to $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}, \mu_N]\{\mathfrak{ab}\}$, which is equipped with a canonical Γ-level structure i, polarization λ, and differential ω.

Let $W_m = \mathcal{W}/p^m\mathcal{W}$, and let R be a W_m-algebra. Write $T_{\alpha/R}$ for the Igusa tower of $M(\mathfrak{c}, \Gamma)$ defined over R. For $f \in H^0(T_{\alpha/R}, \underline{\omega}^k_{/R})$ for a tuple $k \in \mathbb{Z}[I]$, which we can view as a function on the set of quadruples consisting of an AVRM, a polarization, a level structure, and a differential, we can write

$$f = a_{\mathfrak{a},\mathfrak{b}}(0, f) + \sum_{\xi \in (\mathfrak{ab})_+} a_{\mathfrak{a},\mathfrak{b}}(\xi, f) q^\xi.$$

Theorem (q-expansion principle). *With notation as above, let R' be a W_m-subalgebra of R over which T_α is defined, and assume p is unramified in $F/\mathbb{Q}$. Then*

(1) $f = 0$ if and only if $a_{\mathfrak{a},\mathfrak{b}}(\xi, f) = 0$ for every $\xi \in (\mathfrak{ab})_+ \cup \{0\}$;

(2) $f \in H^0(T_{\alpha/R'}, \underline{\omega}^k_{/R'})$ if and only if $a_{\mathfrak{a},\mathfrak{b}}(\xi, f) \in R'$ for all $\xi \in (\mathfrak{ab})_+ \cup \{0\}$.

Proof. The only way that the first statement can fail is if f decomposes into terms from different irreducible components such that the coefficients are not simultaneously zero on all of them; but this is impossible by the irreducibility of T_α. Indeed, the same argument shows that the statement holds for all $f \in H^0(T_{\alpha/R}, \underline{\omega}^k_{/R} \otimes_R M)$.

To see the second statement, observe that we have a short exact sequence

$$0 \to \underline{\omega}^k_{/R'} \to \underline{\omega}^k_{/R} \to \underline{\omega}^k_{/R'} \otimes_R (R/R') \to 0$$

and taking cohomology we therefore have a short exact sequence

$$0 \to H^0(T_{\alpha/R'}, \underline{\omega}^k_{/R'}) \to H^0(T_{\alpha/R}, \underline{\omega}^k_{/R}) \to H^0(T_{\alpha/R'}, \underline{\omega}^k_{/R'} \otimes_R (R/R')).$$

Therefore f, which lives in the middle term, is in the image of the first term if and only if its image vanishes in the third. If $\bar{f}$ is the image of f, we can write it as

$$\bar{f} = b_{\mathfrak{a},\mathfrak{b}}(0, \bar{f}) + \sum_{\xi \in (\mathfrak{ab})_+} b_{\mathfrak{a},\mathfrak{b}}(\xi, \bar{f}) q^\xi$$

for some $b_{\mathfrak{a},\mathfrak{b}}(\xi, \bar{f}) \in M$, and applying the (generalized) first statement with $M = R/R'$ shows that f vanishes in $H^0(T_{\alpha/R'}, \underline{\omega}^k_{/R'} \otimes_R (R/R'))$, and therefore pulls back to $H^0(T_{\alpha/R'}, \underline{\omega}^k_{/R'})$, if and only if all of the $b_{\mathfrak{a},\mathfrak{b}}(\xi, \bar{f})$ vanish. But the $b_{\mathfrak{a},\mathfrak{b}}(\xi, \bar{f})$ are just the images in R/R' of $a_{\mathfrak{a},\mathfrak{b}}(\xi, f)$, and so vanish if and only if the $a_{\mathfrak{a},\mathfrak{b}}(\xi, f)$ are in R'. □

4. Hecke operators

We return to Shimura varieties. Write Sh_K for $\mathrm{Sh}_K(G, X)$. For open compact subgroups $K, K' \subset G(\mathbb{A}_f)$ and $g \in G(\mathbb{A}_f)$, write K^g for $g^{-1}Kg$; we have a projection $p_1 : \mathrm{Sh}_{K^g \cap K'} \to \mathrm{Sh}_{K'}$ induced by the inclusion, i.e. sending $([x, h] \bmod K^g \cap K') \mapsto ([x, h] \bmod K')$, and a second projection $p_g : \mathrm{Sh}_{K^g \cap K'} \to \mathrm{Sh}_K$ sending $([x,h] \bmod K^g \cap K') \mapsto ([x, hg^{-1}] \bmod K)$, which is roughly the projection to K^g twisted by g to land in Sh_K. These projections give an algebraic correspondence $(KgK') := \mathrm{Sh}_{K^g \cap K'} \xrightarrow{p_g \times p_1} \mathrm{Sh}_K \times \mathrm{Sh}_{K'}$ depending only on the double coset.

Now suppose that we have a vector bundle $\mathcal{L}$ on Sh descending to a vector bundle $\mathcal{L}_K$ on Sh_K such that $\pi^*\mathcal{L}_K = \mathcal{L}$ for K sufficiently small, where $\pi : \mathrm{Sh} \to \mathrm{Sh}_K$ is the projection (in practice we will take $\mathcal{L}_K = \underline{\omega}_K^k$), on which an open semi-group $\Delta \subset G(\mathbb{A}_f)$ acts by pullbacks, i.e. the diagram

$$\begin{array}{ccc} \mathcal{L} & \xleftarrow[g^*]{} & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathrm{Sh} & \xrightarrow{g^{-1}} & \mathrm{Sh} \end{array}$$

commutes for $g \in \Delta$ (where we use g^{-1} in the bottom map since the action of $G(\mathbb{A}_f)$ on Sh is a right action and we want a left action on $\mathcal{L}$). We also assume we have trace maps on cohomology

$$\mathrm{Tr}_{K'/K} : H^\bullet(\mathrm{Sh}_K, \mathcal{L}_K) \to H^\bullet(\mathrm{Sh}_{K'}, \mathcal{L}_{K'})$$

for $K \subset K' \subset \Delta$, satisfying the obvious compatibility conditions for towers of subgroups and $\mathrm{Tr}_{K'/K} \circ \mathrm{Res}_{K'/K}$ is multiplication by the degree of $\mathrm{Sh}_K \to \mathrm{Sh}_{K'}$.

The correspondence (KgK') defines the Hecke operator $[KgK'] : H^\bullet(\mathrm{Sh}_K, \mathcal{L}_K) \to H^\bullet(\mathrm{Sh}_{K'}, \mathcal{L}_{K'})$ by

$$[KgK'] = |\det g|_{\mathbb{A}} \cdot \mathrm{Tr}_{K'/(K^g \cap K')} \circ [g] \circ \mathrm{Res}_{K/(K \cap {}^gK')},$$

where ${}^gK' = gK'g^{-1}$ and $[g] : H^\bullet(\mathrm{Sh}_{K \cap {}^gK'}, L_{K \cap {}^gK'}) \to H^\bullet(\mathrm{Sh}_{K^g \cap K'}, \mathcal{L}_{K^g \cap K'})$ is induced by the pullback of the action $g^{-1} : \mathrm{Sh}_{K^g \cap K'} \xrightarrow{\sim} \mathrm{Sh}_{K \cap {}^gK'}$.

In particular, double cosets KgK act on $H^\bullet(\mathrm{Sh}_K, \mathcal{L}_K)$ by Hecke operators. We can define $R(K, \Delta)$ for any compact open subgroup $K \subset \Delta$ of $G(\mathbb{A}_f)$ to be the free $\mathbb{Z}$-module formally generated by double cosets KgK for $g \in \Delta$, with a multiplication which Hida doesn't define and so I won't either; this is the double coset ring for $K \subset \Delta$, and thus acts on $H^\bullet(\mathrm{Sh}_K, \mathcal{L}_K)$.