Λ-adic Galois representations

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Abstract. Following [4, chapter 7, section 5], we show that to each ordinary $\mathcal{I}$-adic form we can associate a unique Galois representation over K satisfying certain good properties, where K is a finite extension of $\operatorname{Frac}\Lambda$ and $\mathcal{I}$ is the integral closure of Λ in K.

1. Introduction

As previously, fix a finite extension E of $\mathbb{Q}_p$ (let's assume $p > 2$ for simplicity) with ring of integers $\mathcal{O}$, and let $\Lambda = \mathcal{O}[(1+p\mathbb{Z}_p)^\times] \simeq \mathcal{O}[\mathbb{Z}_p] \simeq \mathcal{O}[[T]]$. Let K be a finite extension of $\operatorname{Frac}\Lambda$ and define $\mathcal{I}$ to be the integral closure of Λ in K, with maximal ideal $\mathfrak{m}$. Fix a topological generator $u = 1+p$ of $(1+p\mathbb{Z}_p)^\times$, and define the map $\kappa : (1+p\mathbb{Z}_p)^\times \to \Lambda^\times$ defined on powers of u by $\kappa(u^n) = (1+X)^n$ and extended to all of $1+p\mathbb{Z}_p$ by continuity.

We say that a Galois representation $\pi : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(K)$ is continuous if there exists an $\mathcal{I}$-submodule L of K^2 such that $L \otimes_{\mathcal{I}} K = K^2$, L is stable under π, and the restriction $\pi : G_{\mathbb{Q}} = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{End}_{\mathcal{I}}(L)$ is continuous with respect to the $\mathfrak{m}$-adic topology on L. This definition is independent of the choice of L.

Why this definition of continuity? We could instead take the topology coming from a topology on K, but since $\mathcal{I}$ has Krull dimension 2 K cannot be locally compact; but since $G_{\mathbb{Q}}$ is profinite and therefore compact under the Krull topology, its image under a continuous representation is compact, and therefore gives us only a very small portion of $\operatorname{GL}_2(K)$. On the other hand, we can find some n such that $\mathcal{I}^n$ surjects onto L; thus each $L/\mathfrak{m}^iL$ is the image of $(\mathcal{I}/\mathfrak{m}^i)^n$, which is finite (think for example of $\mathcal{I} = \Lambda$) so that the induced topology on $\operatorname{End}_{\mathcal{I}}(L)$ makes it compact. Therefore continuity with respect to this topology is more natural for Galois representations.

We say that π is unramified at a prime ℓ if the inertia group at ℓ is in the kernel of π.

We didn't get to this last time, but in Hung's notes [1] there's a duality result that we'll use: I'll just give the statement, and you can see there for the proof.

Let χ be a Dirichlet character modulo p, and for any Λ-algebra A let $h^{\mathrm{ord}}(\chi; A) = \operatorname{End}_A(\mathbb{S}^{\mathrm{ord}}(\chi; A))$. For any modular form f write $a_1(f)$ for its first Fourier coefficient.

Proposition 1.1. *The pairing $(h, f) \mapsto a_1(h(f))$ defines a perfect pairing between $h^{\mathrm{ord}}(\chi; A)$ and $\mathbb{S}^{\mathrm{ord}}(\chi; A)$. In particular $\operatorname{Hom}_A(h^{\mathrm{ord}}(\chi; A), A) \simeq \mathbb{S}^{\mathrm{ord}}(\chi; A)$, and $\varphi \in \operatorname{Hom}_A(h^{\mathrm{ord}}(\chi; A))$ is a homomorphism of Λ-algebras if and only if the corresponding cusp form is a normalized Hecke eigenform with coefficients in A.*

Let $F \in \mathbb{S}^{\mathrm{ord}}(\chi, \mathcal{I})$ be a normalized eigenform. Since $\mathcal{I}$ is a Λ-algebra, F corresponds to a unique homomorphism of Λ-algebras $\lambda : h^{\mathrm{ord}}(\chi; \mathcal{I}) \to \mathcal{I}$. Our main goal for today is to prove the following result.

Theorem 1.2. *There exists a unique Galois representation $\pi : G_{\mathbb{Q}} \to \operatorname{GL}_2(K)$ such that*

(*i*) *π is continuous and absolutely irreducible;*

(*ii*) *π is unramified at each prime $\ell \neq p$;*

(iii) for each prime $\ell \neq p$, we have

$$\det(1 - \pi(\mathrm{Frob}_\ell)T) = 1 - \lambda(T_\ell)T + \chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1}T^2$$

where T_ℓ is the Hecke operator at ℓ and $\langle \ell \rangle$ is the Diamond operator.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{I}$, and π be a Galois representation as in Theorem 1.2. Let $k_\mathfrak{p} = \mathrm{Frac}(\mathcal{I}/\mathfrak{p})$, and for each element $t \in \mathcal{I}$ write $t(\mathfrak{p})$ for the image of t under the surjection $\mathcal{I} \to \mathcal{I}/\mathfrak{p}$. We want to reduce π modulo $\mathfrak{p}$; this reduction should be a representation $\pi_\mathfrak{p} : G_\mathbb{Q} \to \mathrm{GL}_2(k_\mathfrak{p})$ satisfying

(a) $\pi_\mathfrak{p}$ is continuous and semisimple;
(b) $\pi_\mathfrak{p}$ is unramified at each prime $\ell \neq p$;
(c) for each prime $\ell \neq p$, we have

$$\det(1 - \pi_\mathfrak{p}(\mathrm{Frob}_\ell)T) = 1 - \lambda(T_\ell)(\mathfrak{p})T + (\chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1})(\mathfrak{p})T^2.$$

Note that in particular if $\mathfrak{p}$ is the kernel of the specialization $X \mapsto \epsilon(u)u^k - 1$ for some character ϵ modulo p then we have $(\chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1})(\mathfrak{p}) = \chi(\ell)\ell^{k-1}$ and $\lambda(T_\ell)(\mathfrak{p})$ is the ℓth Fourier coefficient of the reduction of F modulo $\mathfrak{p}$, so these are the expected Galois representations corresponding to F modulo $\mathfrak{p}$. Thus we can think of Theorem 1.2 as providing a way of deforming Galois representations coming from p-adic modular forms.

The continuity of $\pi_\mathfrak{p}$ is defined similarly to above: if L is an $\mathcal{I}$-submodule of K^2 with respect to which π is continuous, then $L/\mathfrak{p}L$ is a submodule of $k_\mathfrak{p}^2$ with respect to which we want $\pi_\mathfrak{p}$ to be continuous in the induced $\mathfrak{m}$-adic topology. (Since $\mathcal{I}$ has Krull dimension 2, each $k_\mathfrak{p}$ is locally compact, and so in this case we can think equivalently of the natural topology on $\mathrm{GL}_2(k_\mathfrak{p})$ coming from the topology on $k_\mathfrak{p}$.)

We say that any Galois representation $G_\mathbb{Q} \to \mathrm{GL}_2(\overline{k}_\mathfrak{p})$ is *residual* at $\mathfrak{p}$ if it satisfies these properties.

Since L need not be free, it is not a priori obvious that we can find such a residual representation, but in fact it is true:

Proposition 1.3. *Let π be a Galois representation as in Theorem 1.2. Then for every prime ideal $\mathfrak{p}$ there exists a residual representation $\pi_\mathfrak{p}$ at $\mathfrak{p}$, unique up to $\overline{k}_\mathfrak{p}$-isomorphisms.*

Proof sketch. The idea is to replace L by a free module V of rank 2 over an $\mathcal{I}$-algebra A; we can take A to be the localization of $\mathcal{I}$ at $\mathfrak{p}$ and $V = L \otimes_\mathcal{I} A$, for $\mathfrak{p}$ of height 1 to ensure good properties of A. Then we can reduce the restriction of π to $\mathrm{GL}(V) \simeq \mathrm{GL}_2(A)$ modulo $\mathfrak{p}$ to get a representation satisfying the desired properties; and we repeat the process to get the result for primes of all heights. □

Of course, we might expect to find representations $G_\mathbb{Q} \to \mathrm{GL}_2(k_\mathfrak{p})$ at a prime $\mathfrak{p}$ of $\mathcal{I}$ satisfying conditions (a, b, c), regardless of the existence of any "global" representation. To prove Theorem 1.2, it is natural to ask if we can go the other way, local-global style: given sufficiently many representations $\pi_\mathfrak{p} : G_\mathbb{Q} \to \mathrm{GL}_2(k_\mathfrak{p})$ residual at $\mathfrak{p}$, can we somehow glue them together to form a representation $\pi : G_\mathbb{Q} \to \mathrm{GL}_2(K)$ satisfying the conditions of Theorem 1.2?

Answer: yes! And this will be our main tool in proving Theorem 1.2, as summarized in the following theorem of Wiles [7].

Theorem 1.4. *Let $F \in \mathbb{S}^{\mathrm{ord}}(\chi, \mathcal{I})$ be a normalized eigenform corresponding to the Λ-algebra homomorphism $\lambda : h^{\mathrm{ord}}(\chi; \mathcal{I}) \to \mathcal{I}$, and for each prime $\mathfrak{p}$ of $\mathcal{I}$ write $\mathcal{O}_\mathfrak{p}$ for the ring of integers of $k_\mathfrak{p}$. If there are infinitely many primes $\mathfrak{p}$ such that there exists a representation $\pi_\mathfrak{p} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_p)$ residual at $\mathfrak{p}$, then there exists a representation $\pi : G_\mathbb{Q} \to \mathrm{GL}_2(K)$ satisfying the conditions of Theorem 1.2.*

Note that the condition that the image of $\pi_\mathfrak{p}$ lie in $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) \subset \mathrm{GL}_2(k_\mathfrak{p})$ is not a serious one: although this is not necessarily true of an arbitrary representation, any continuous representation (of a compact group) into $\mathrm{GL}_2(k_\mathfrak{p})$ is conjugate to one valued on $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$. Indeed, it is enough to find an invariant lattice, so that changing to the corresponding basis gives a representation valued in $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$; if L_0 is any $\mathcal{O}_\mathfrak{p}$-lattice in $k_\mathfrak{p}^2$, then $\pi_\mathfrak{p}^{-1}(\mathrm{GL}(L_0))$ is a finite index subgroup since $\mathrm{GL}(L_0)$ is open in $\mathrm{GL}_2(k_\mathfrak{p})$, and so we can sum over finitely many cosets $\sigma_i \in G_\mathbb{Q}/\pi_\mathfrak{p}^{-1}(\mathrm{GL}(L_0))$ to get an invariant lattice $\sum_i \sigma_i L_0$.

Given this theorem, we still need the input of an infinite family of suitable Galois representations. This is provided by the following theorem.

Theorem 1.5. *Let k be a positive integer, χ be a Dirichlet character modulo N, M be a finite extension of $\mathbb{Q}_p$, and $\lambda : h_k(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \to M$ be a homomorphism. Then there exists a unique representation $\pi : G_\mathbb{Q} \to \mathrm{GL}_2(M)$ such that*

(*i*) *π is continuous and absolutely irreducible over M;*

(*ii*) *π is unramified at each prime ℓ not dividing Np;*

(*iii*) *for each prime ℓ not dividing Np, we have*

$$\det(1 - \pi(\mathrm{Frob}_\ell)T) = 1 - \lambda(T_\ell)T + \chi(\ell)\ell^{k-1}T^2.$$

This is due to Eichler–Shimura [6] and Igusa [5] in the case $k = 2$, Deligne [2] for $k > 2$, and Deligne–Serre [3] for $k = 1$.

For each k and ϵ as above, we have a specialization map with kernel some prime ideal $\mathfrak{p}$; Theorem then gives at each such $\mathfrak{p}$ a residual representation with respect to F by taking the specialization of the corresponding map λ. Thus with the input of Theorem 1 we've reduced Theorem 1.2 to Theorem 1.4.

The remainder of today will therefore be about proving Theorem 1.4. We'll do this by introducing a notion of pseudo-representations and showing that they satisfy properties which will allow us to do a local-to-global–type construction on them, and that we can use this to produce an honest representation satisfying the desired conditions.

2. Reduction to pseudo-representations

Fix a prime $\mathfrak{p}$ of $\mathcal{I}$, and let $\pi : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ be residual at $\mathfrak{p}$ with respect to some $F \in \mathbb{S}^{\mathrm{ord}}(\chi; \mathcal{I})$ and its corresponding homomorphism $\lambda : h^{\mathrm{ord}}(\chi; \mathcal{I}) \to \mathcal{I}$. Let $\mathbb{Q}^{\mathrm{unr},p}$ be the maximal extension of $\mathbb{Q}$ unramified away from p, and set $G = \mathrm{Gal}(\mathbb{Q}^{\mathrm{unr},p}/\mathbb{Q})$; since π is unramified away from p, it factors through the restriction $\pi : G_\mathbb{Q} \twoheadrightarrow G \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ and so we can consider π to be a representation of G.

Let $L = \mathcal{O}_\mathfrak{p}^2$, viewed as a G-module via π. Let $c \in G$ be complex conjugation; in $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ it acts by -1, and so $\det \pi(c) = \chi(-1)^k(-1)^{k-1} = -1$. Therefore the eigenvalues of $\pi(c)$ are ± 1, since $c^2 = 1$, and so we can decompose L into the eigenspaces $L_+ \oplus L_-$.

Writing π in the corresponding basis, we have

$$\pi(c) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Define functions $a, b, c, d : G \to \mathcal{O}_\mathfrak{p}$ such that

$$\pi(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

for each $\sigma \in G$. Define $x : G \times G \to \mathcal{O}_\mathfrak{p}$ by $x(\sigma, \tau) = b(\sigma)c(\tau)$. Each of a and d is continuous on G, since

$$\operatorname{Tr}\pi(\sigma) = a(\sigma) + d(\sigma), \qquad \operatorname{Tr}\pi(c\sigma) = \operatorname{Tr}(\pi(c)\pi(\sigma)) = a(\sigma) - d(\sigma)$$

are both continuous on G; since π is a group homomorphism,

$$\pi(\sigma\tau) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix} = \begin{pmatrix} a(\sigma\tau) & b(\sigma\tau) \\ c(\sigma\tau) & d(\sigma\tau) \end{pmatrix},$$

and writing this out gives among other things $a(\sigma)a(\tau) + b(\sigma)c(\tau) = a(\sigma)a(\tau) + x(\sigma,\tau) = a(\sigma\tau)$, so the continuity of x follows from the continuity of a. Using the definition of x and the product formula above, we also have the following properties:

(a) $a(\sigma\tau) = a(\sigma)a(\tau) + x(\sigma,\tau)$, $d(\sigma\tau) = d(\sigma)d(\tau) + x(\tau,\sigma)$, and

$$x(\sigma\tau, \sigma'\tau') = a(\sigma)a(\tau')x(\tau,\sigma') + a(\tau')d(\tau)x(\sigma,\sigma') + a(\sigma)d(\sigma')x(\tau,\tau') + d(\tau)d(\sigma')x(\sigma,\tau');$$

(b) $a(1) = d(1) = 1$, $a(c) = -d(c) = 1$, and $x(\sigma,1) = x(\sigma,c) = x(1,\sigma) = x(c,\sigma) = 0$;

(c) $x(\sigma,\tau)x(\sigma',\tau') = x(\sigma,\tau')x(\sigma',\tau)$.

For any (commutative) topological ring R and continuous functions $a, d : G \to R$ and $x : G \times G \to R$, we say that a triple $\pi' = (a, d, x)$ is a *pseudo-representation* of G if a, d, and x satisfy conditions (a, b, c) above. In this case we define the trace

$$\operatorname{Tr}\pi' = a(\sigma) + d(\sigma)$$

and the determinant

$$\det\pi'(\sigma) = a(\sigma)d(\sigma) - x(\sigma,\sigma).$$

It is clear from the definition and our calculations above that every continuous representation $G \to \mathrm{GL}_2(R)$ for a topological ring R yields a unique pseudo-representation.

There are two main propositions about pseudo-representations that we'll need; together these will be enough to prove Theorem 1.4.

Proposition 2.1. *Let R be a topological integral domain with fraction field M, and suppose that $\pi' = (a, d, x)$ is a pseudo-representation $G \to R$. Then there exists a continuous representation $\pi : G \to \mathrm{GL}_2(M)$ with $\operatorname{Tr}\pi = \operatorname{Tr}\pi'$ and $\det\pi = \det\pi'$.*

Proposition 2.2. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $\mathcal{I}$, and let $\pi_\mathfrak{a}$ and $\pi_\mathfrak{b}$ be pseudo-representations of G into $\mathcal{I}/\mathfrak{a}$ and $\mathcal{I}/\mathfrak{b}$ respectively, which are compatible in the sense that there exists a dense subset Σ of G and functions $T, D : \Sigma \to \mathcal{I}/\mathfrak{a} \cap \mathfrak{b}$ such that for every $\sigma \in \Sigma$ we have*

$$\operatorname{Tr} \pi_\mathfrak{a}(\sigma) \equiv T(\sigma) \pmod{a}, \qquad \operatorname{Tr} \pi_\mathfrak{b}(\sigma) \equiv T(\sigma) \pmod{\mathfrak{b}}$$

and

$$\det \pi_\mathfrak{a}(\sigma) \equiv D(\sigma) \pmod{a}, \qquad \det \pi_\mathfrak{b}(\sigma) \equiv D(\sigma) \pmod{\mathfrak{b}}.$$

Then there exists a pseudo-representation $\pi_{\mathfrak{a}\cap\mathfrak{b}} : G \to \mathcal{I}/\mathfrak{a} \cap \mathfrak{b}$ such that for every $\sigma \in \Sigma$ we have

$$\operatorname{Tr} \pi_{\mathfrak{a}\cap\mathfrak{b}}(\sigma) = T(\sigma), \qquad \det \pi_{\mathfrak{a}\cap\mathfrak{b}}(\sigma) = D(\sigma).$$

We now prove Theorem 1.4, assuming these two propositions. By assumption we have infinitely many primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ of $\mathcal{I}$ at which we have residual representations $\pi_\mathfrak{p}$ with respect to our eigenform F; each is also a pseudo-representation. By (the infinite version of) Chebotarev's density theorem, since $\mathbb{Q}^{\mathrm{unr},p}$ is unramified away from p the Frobenius elements Frob_ℓ for $\ell \neq p$ form a dense subset Σ of $G = \mathrm{Gal}(\mathbb{Q}^{\mathrm{unr},p}/\mathbb{Q})$; since $\pi_\mathfrak{p}$ is residual, we have

$$\operatorname{Tr} \pi_\mathfrak{p}(\mathrm{Frob}_\ell) = \lambda(T_\ell)(\mathfrak{p})$$

and

$$\det \pi_\mathfrak{p}(\mathrm{Frob}_\ell) = (\chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1})(\mathfrak{p}).$$

Therefore, applying Proposition 2.2 at $\mathfrak{p}_1, \mathfrak{p}_2$ we have a pseudo-representation $\pi_{\mathfrak{p}_1\cap\mathfrak{p}_2} : G \to \mathcal{I}/\mathfrak{p}_1 \cap \mathfrak{p}_2$ such that

$$\operatorname{Tr} \pi_{\mathfrak{p}_1\cap\mathfrak{p}_2}(\mathrm{Frob}_\ell) \equiv \lambda(T_\ell) \pmod{\mathfrak{p}_1 \cap \mathfrak{p}_2}$$

and

$$\det \pi_{\mathfrak{p}_1\cap\mathfrak{p}_2}(\mathrm{Frob}_\ell) \equiv \chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1} \pmod{\mathfrak{p}_1 \cap \mathfrak{p}_2}.$$

Repeating with $\mathfrak{p}_1 \cap \mathfrak{p}_2$ and $\mathfrak{p}_3$, we similarly get a pseudo-representation $\pi_{\mathfrak{p}_1\cap\mathfrak{p}_2\cap\mathfrak{p}_3}$; iterating, for every $n \geq 1$ we have a pseudo-representation $\pi_n : G \to \mathcal{I}/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ satisfying

$$\operatorname{Tr} \pi_n(\mathrm{Frob}_\ell) \equiv \lambda(T_\ell) \pmod{\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n}$$

and

$$\det \pi_n(\mathrm{Frob}_\ell) \equiv \chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1} \pmod{\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n},$$

and at each n we have $\operatorname{Tr} \pi_n \equiv \operatorname{Tr} \pi_{n-1} \pmod{\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{n-1}}$ on each Frob_ℓ and similarly for the determinant; since the Frob_ℓ are dense in G and both sides are continuous, this is true on all of G, so that we have an inverse system. Taking the limit as $n \to \infty$ gives a pseudo-representation $\pi' := \varprojlim_n \pi_n : G \to \varprojlim_n \mathcal{I}/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = \mathcal{I}$ with trace $\lambda(T_\ell)$ and determinant $\chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1}$ at Frob_ℓ; applying Proposition 2.1 with $R = \mathcal{I}$ then gives a genuine representation $G \to \mathrm{GL}_2(K)$ which, upon composing with the restriction $G_\mathbb{Q} \twoheadrightarrow G$, satisfies the conditions of Theorem 1.2 as desired.

It remains only to prove these two propositions, which we will now do.

3. Proof of Proposition 2.1

Let R be a topological integral domain with fraction field M and $\pi' = (a, d, x)$ be a pseudo-representation $G \to R$. There are two cases: either x is identically zero or it is not.

First, suppose that x is identically zero, so that the condition that π' is a pseudo-representation implies that $a(\sigma\tau) = a(\sigma)a(\tau)$ and similarly for d. Then setting

$$\pi(\sigma) = \begin{pmatrix} a(\sigma) & \\ & d(\sigma) \end{pmatrix}$$

is an honest representation $G \to \mathrm{GL}_2(M)$, and since a and d are continuous so is π; and it manifestly has the same trace and determinant as π'.

Therefore suppose that we can find some σ_0, τ_0 such that $x(\sigma_0, \tau_0) \neq 0$. Then we define functions $b, c : G \to R$ by $b(\sigma) = \frac{x(\sigma,\tau_0)}{x(\sigma_0,\tau_0)}$ and $c(\sigma) = x(\sigma_0, \sigma)$ for each $\sigma \in G$. These are continuous since x is. Since π' is a pseudo-representation, we have $b(\sigma)c(\tau) = \frac{x(\sigma,\tau_0)}{x(\sigma_0,\tau_0)} x(\sigma_0, \tau) = x(\sigma, \tau)$; similarly the various conditions on π' work out to imply that if we define

$$\pi(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

then

$$\pi(1) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad \pi(c) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

and

$$\pi(\sigma\tau) = \pi(\sigma)\pi(\tau).$$

Therefore π is a continuous representation $G \to \mathrm{GL}_2(K)$ with the same trace and determinant as π'.

4. Proof of Proposition 2.2

Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $\mathcal{I}$. By the (rather generalized) Chinese remainder theorem we have a short exact sequence of $\mathcal{I}$-modules

$$0 \to \mathcal{I}/\mathfrak{a} \cap \mathfrak{b} \xrightarrow{i} \mathcal{I}/\mathfrak{a} \oplus \mathcal{I}/\mathfrak{b} \xrightarrow{p} \mathcal{I}/(\mathfrak{a} + \mathfrak{b}) \to 0$$

where the injection is $i : x \mapsto (x \bmod \mathfrak{a}, x \bmod b)$ and the surjection is $p : (x, y) \mapsto x - y \bmod \mathfrak{a} + \mathfrak{b}$. Letting $\pi_\mathfrak{a}$ and $\pi_\mathfrak{b}$ be the given pseudo-representations of G into $\mathcal{I}/\mathfrak{a}$ and $\mathcal{I}/\mathfrak{b}$ respectively, define $\pi = \pi_\mathfrak{a} \oplus \pi_b$ to be a pseudo-representation $G \to \mathcal{I}/\mathfrak{a} \oplus \mathcal{I}/\mathfrak{b}$. For $\sigma \in \Sigma$, we have $\operatorname{Tr} \pi(\sigma) = \operatorname{Tr} \pi_\mathfrak{a}(\sigma) + \operatorname{Tr} \pi_\mathfrak{b}(\sigma) \in \mathcal{I}/\mathfrak{a} \oplus \mathcal{I}/\mathfrak{b}$; by assumption this is the image of $T(\sigma)$ under i, and therefore $p(\operatorname{Tr} \pi(\sigma)) = 0$ for every $\sigma \in \Sigma$. Since Σ is dense in G and $p \circ \operatorname{Tr} \pi$ is continuous, $p(\operatorname{Tr} \pi(\sigma)) = 0$ for every $\sigma \in G$, and so $\operatorname{Tr} \pi(\sigma)$ is always in the image of i. Writing $\pi = (a, d, x)$, we can reconstruct each of a, d, and x from $\operatorname{Tr} \pi$ by

$$a(\sigma) = \frac{\operatorname{Tr} \pi(\sigma) + \operatorname{Tr}(\pi(c\sigma))}{2}, \qquad d(\sigma) = \frac{\operatorname{Tr} \pi(\sigma) - \operatorname{Tr} \pi(c\sigma)}{2}, \qquad x(\sigma, \tau) = a(\sigma\tau) - a(\sigma)a(\tau),$$

so since each of $\operatorname{Tr}\pi(\sigma)$ and $\operatorname{Tr}\pi(c\sigma)$ is in the image of i so are each of $a(\sigma)$, $d(\sigma)$, and $x(\sigma,\tau)$ (recalling that 2 is invertible in $\mathcal{I}$). Therefore π is a pseudo-representation of G into the image of i, and so taking the preimage under i we get a pseudo-representation π' of G into $\mathcal{I}/\mathfrak{a}\cap\mathfrak{b}$ satisfying $\operatorname{Tr}\pi'(\sigma) = T(\sigma)$ for $\sigma \in \Sigma$, and

$$\det \pi'(\sigma) = i^{-1}(a(\sigma)d(\sigma) - x(\sigma,\sigma)) = \frac{\operatorname{Tr}\pi'(\sigma)\operatorname{Tr}\pi'(c\sigma)}{2} - \frac{\operatorname{Tr}\pi'(c\sigma^2)}{2}$$

must also agree with $\det \pi_{\mathfrak{a}}$ and $\det \pi_{\mathfrak{b}}$ modulo $\mathfrak{a}$ and $\mathfrak{b}$ respectively since the trace does.

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