Dual groups of spherical varieties

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1. Summary of Sug Woo Shin's talks

The "basic" setup is: let G be a (split) reductive group, $H \subset G$ a subgroup, and consider the G-variety $X = H\backslash G$ over a field F. More generally, we could consider any normal quasi-affine G-variety X on which there is an open dense B-orbit; such varieties are called spherical.

When F is a global field we can formulate the periods and L-functions fairly easily: if Φ is a suitable Schwartz function on $X(\mathbb{A}_F)$, then we can define

$$\Theta_\Phi(g) = \sum_{x \in X(F)} \Phi(xg)$$

on $[G] = G(F)\backslash G(\mathbb{A}_F)$, and then define the X-period by pairing $\varphi \in L^2([G])$ with Θ_Φ,

$$\langle \varphi, \Theta_\Phi \rangle = \int_{[G]} \varphi(g)\Theta_\Phi(g)\, dg.$$

In the special case where $X = H\backslash G$, if we define

$$P_H(\varphi)(x) = \int_{[H]} \varphi(hx)\, dh,$$

then

$$\int_X P_H(\varphi)(x)\Phi(x)\, dx = \int_{[H\backslash G]} \Phi(x) \int_{[H]} \varphi(hx)\, dh\, dx = \int_{[G]} \Theta_\Phi(g)\varphi(g) = \langle \varphi, \Theta_\Phi \rangle$$

so we recover a Φ-weighted H-period. A natural question on this side is: for what φ (i.e. appearing in which subrepresentations π) is this X-period nonzero? Another: per the conjectures that Gyujin sketched last time, this should correspond to some L-value; how do we construct it? The L-value should have an Euler factorization, so ideally so too should the period; how do we get this?

Now say everything is over a local field F. Then $L^2(X(F))$ or $C^\infty(X(F))$ carry an action of $G(F)$ and so decompose as a sum (integral) of automorphic representations of $G(F)$. Spherical varieties in particular have the following good property: for any admissible irreducible representation π of $G(F)$, the multiplicity $\dim \mathrm{Hom}_{G(F)}(\pi, C^\infty(X(F)))$ is finite. One can think of spherical varieties as generalizations of toric varieties to nonabelian groups.

The main question this construction raises is: which π appear in $C^\infty(X(F))$? (and perhaps secondarily what are their multiplicities?)

The local Langlands program tells us that such representations π should be classified by Weil–Deligne representations $\rho_\pi : W_F \times \mathrm{SL}_2 \to \check{G} \rtimes W_F$ (we'll neglect the W_F-factor on the right). The idea is to construct a group $\check{G}_X$ together with a map $\iota_X : \check{G}_X \times \mathrm{SL}_2 \to \check{G}$. Roughly speaking, π should appear in $C^\infty(X(F))$ if and only if ρ_π factors through ι_X, i.e. there is a map $\phi_\pi : W_F \to \check{G}_X$ such that ρ_π is the composition $W_F \times \mathrm{SL}_2 \xrightarrow{\phi_\pi \times \mathrm{id}} \check{G})X \times \mathrm{SL}_2 \xrightarrow{\iota_X} \check{G}$; a more precise formulation of this I imagine will occur in the next talk.

Also in the next talk (?), we'll see how the numerical picture falls out of this setup: roughly speaking we get a local L-factor from ρ_π and a local X-period from π, and when π occurs in $L^2(X(F))$ then the local X-period is nonzero and the local L-factor comes from ϕ_π. These then give Euler factorizations of the global periods and L-values.

With the data of $\check{G}_X$ and ι_X in hand, as well as a certain $\check{G}_X$-representation V_X, we can now answer some of the questions in the global case: again, an automorphic representation π should have nonzero X-period if and only if its Arthur parameter factors through ϕ_π and ι_X; and the corresponding L-value should be $\frac{L(\phi_\pi, V_X)}{L(\phi_\pi, \mathrm{Ad})}$.

The most classical case is the "group case" where $G = H \times H$ and $H \hookrightarrow H \times H$ is the diagonal embedding. Then $X = H \backslash G \simeq H$, with the action of $H \times H$ by $(h_1, h_2) \cdot h = h_2^{-1} h h_1$. In this case Θ_Φ is the usual theta function for H. We have $\check{G}_X = \check{H} \subset \check{G} = \check{H} \times \check{H}$; it turns out that this is no longer the diagonal embedding, but (id, c) for c the Chevalley involution; we'll discuss this case a little more in the second section of the talk. The map from SL_2 is trivial in this case. So an H-distinguished automorphic form of $H \times H$ is one whose L-parameter factors through $\check{H} \xrightarrow{\mathrm{id}, c} \check{H} \times \check{H}$; twisting an L-parameter ρ_π by c should give roughly the L-parameter of the contragradient $\rho_{\pi^\vee}$, so the distinguished representations $\pi_1 \otimes \pi_2$ should be the $\pi \otimes \pi^\vee$.

The other simple homogeneous case to work through is $X = G \backslash G = *$. In this case $\Theta_\Phi = \Phi$, so the pairing is just integrating against Schwartz functions. We'll see in the next section that $\check{G}_X$ is trivial; this is perhaps unsurprising when X is trivial. The SL_2-map is principal and so forms all of ι_X.

A less trivial homogeneous case is the "Hecke period" $G = \mathrm{PGL}_2$ and $H = \mathbb{G}_\mathrm{m}$, embedded as the top-left coordinate. This gives rise to the Hecke period $\int_{[H]} \varphi(h)\, dh$, which Waldspurger's formula tells us is supposed to be roughly $L(1/2, \pi)$; we could also twist by inserting a factor of $|h|^s$, which would shift the L-value to $L(1/2 + s, \pi)$. In this case, $\check{G}_X$ should again be $\mathbb{G}_\mathrm{m}$, embedded similarly into $\check{G} = \mathrm{SL}_2$. Through the formalism of BZSV, which generally we won't discuss until future talks, we should be able to match up these sorts of examples with duals; the dual in this case is $\check{G} = \mathrm{SL}_2$ acting on its standard representation $\mathbb{A}^2$.

Finally, we include an example of a non-homogeneous spherical variety: $X = \mathbb{A}^1$ with the natural action of $G = \mathbb{G}_\mathrm{m}$. In this case an automorphic form φ on $[G]$ is just a Hecke character χ, and its X-period is

$$\int_{F^\times \backslash \mathbb{A}_F^\times} \chi(g) \Theta_\Phi(g)\, dg = \int_{F^\times \backslash \mathbb{A}_F^\times} \chi(g) \int_F \Phi(xg)\, dx\, dg = \Phi(0) \int_{[\mathbb{G}_\mathrm{m}]} \chi(g)\, dg + \int_{\mathbb{A}_F^\times \chi(g)\Phi(g)\, dg}$$

and by Tate's thesis the second term is (for a suitable choice of Φ) essentially the L-value $L(0, \chi)$. Here $\check{G}_X = \mathbb{G}_\mathrm{m} = \check{G}$ with identity map.

2. Construction of the dual group

Shin omits the details of the construction, and for good reason: in general it is quite complicated. We will try to flesh it out a little though beyond the material of the videos.

The main idea is this: first, via a classification theorem, we can associate to a spherical G-variety X a root datum Φ_X. We could then construct an algebraic group G_X with this root datum, and hope for a map $G_X \to G$ to relate G_X-representations to G-representations; but it turns out that such a map does not always exist. Instead, the idea of Gaitsgory–Nadler is to look at the Langlands duals, and hope for a map $\check{G}_X \to \check{G}$; then we can study the same problem as an instance of Langlands functoriality. Of course we've also seen that the Arthur parameters for $\check{G}_X$ naturally govern the X-distinguishedness of G-representations, making this idea concrete in a way.

This gives the construction of $\check{G}_X$, as the algebraic group with root system $\Phi_X^\vee$. What remains unclear is the map $\check{G}_X \to \check{G}$, which in fact should extend to $\check{G}_X \times \mathrm{SL}_2 \to \check{G}$.

If $A \subset B \subset G$ is the maximal torus and $P_X \supset B$ is the stabilizer of the open B-orbit $X^\circ \subset X$ with unipotent radical N_X, the action of the Levi subgroup $L_X = P_X/N_X$ acts on X°/N_X through the faithful action of a torus quotient A_X, giving a composition

$$A \hookrightarrow B \hookrightarrow P_X \twoheadrightarrow L_X \twoheadrightarrow A_X$$

which turns out to be surjective. Taking duals then gives an injection $A_X^* \hookrightarrow A^*$. As A is our fixed maximal torus for G, A^* is the maximal torus for $\check{G}$; the group $\check{G}_X$ should have maximal torus A_X^*, with $A_X^* \to A^*$ extending to $\check{G}_X \to \check{G}$. The existence of this extension is actually conjectural in SV, but is proven by Knop–Schalke; we will otherwise neglect it and focus on the description of the group $\check{G}_X$. (The extension to SL_2 is by computing the centralizer of the image of $\check{G}_X$, which we will also neglect.)

One can study spherical varieties via combinatorial data. In particular, we can embed them into "wonderful varieties," i.e. proper smooth G-varieties with an open G-orbit whose complement is the union of finitely many irreducible divisors $D_1, \ldots, D_r$ with certain technical properties. One can then proceed via two steps: first, we can classify wonderful varieties by certain combinatorial data (spherical G-systems); classify G-equivariant open embeddings $X \hookrightarrow \overline{X}$ of spherical varieties into wonderful varieties; and combine these to produce a combinatorial description of spherical varieties.

However, we don't actually need our combinatorial data to determine the spherical variety; so a weaker collection of data suffices, which in fact we can associate to any G-variety. This is a *weak spherical datum*: it consists of a subgroup $\Xi \subset X^*(G)$ of the character group together with two finite subsets $\Sigma \subset \Xi$, $S^p \subset \Xi$ satisfying certain compatibility relations, designed to mimic the properties of the data which do in fact come from spherical varieties. These are produced as follows:

Let $k(X)^{(B)} \subset k(X)^*$ be the subgroup consisting of rational functions on X such that there exists $\chi_f \in X^*(G)$ such that $b \cdot f = \chi_f(b) f$ all $b \in B$. This produces a map $k(X)^{(B)} \to X^*(G)$ sending $f \mapsto \chi_f$; its image is our subgroup Ξ.

In fact, the "multiplicity free" property of spherical varieties is that the space of f with $\chi_f = \chi$ for any fixed χ is one-dimensional, so up to a scalar we can also associate to χ a function f_χ. This lets us view the space W of G-invariant discrete valuations v on X as living inside Ξ^*: for every $\chi \in \Xi$, fix f_χ and set $\langle v, \chi \rangle = v(f_\chi)$. Then we can fix a set $\Sigma \subset \Xi$ such that $W \subset \Xi^*$ can be written as $\{v \in \Xi^* | \langle v, \sigma \rangle \geq 0 \forall \sigma \in \Sigma\}$.

Finally, we define $S^p \subset \Xi$ as the fixed simple roots for G inside Ξ spanning the root system for the Levi L_X; this is not as key for our construction so we won't elaborate. One can show various properties of (Ξ, Σ, S^p), e.g. that Σ is linearly independent; the key fact is that $(\Xi, \Sigma, \Xi^\vee, \Sigma^\vee)$ is a root datum Φ_X when X is spherical. In particular, we can take the dual root datum $\Phi_X^\vee$ to construct the algebraic group with this root datum $\check{G}_X$.

By considering the divisors in the complement of the open orbit, we can take the discrete valuations associated to them, which as above gives elements of Ξ^*, among which we can take the dominant coroots; then in nice cases we define the $\check{G}_X$-representation S_X to be the representation with highest weights given by these coroots. (In general we need a more complicated construction.) At least conjecturally this representation is symplectic.

What is of more interest is the representation $V_X = S_X \oplus (\mathfrak{g}_X^\perp \cap \check{\mathfrak{g}}_e)$ where $\check{\mathfrak{g}}_e$ is apparently the centralizer of a principal nilpotent e; hopefully the naturality of this definition will become clearer later on, but it is (apparently) the right thing to plug into our L-value machine.

We can now return to some of our examples earlier to justify our claims about $\check{G}_X$.

In the group case $G = H \times H$, the Borel subgroup is $B \times B$ for B a Borel of H. Its open dense orbit on $H\backslash G \simeq H$ is BwB for w (a lift of) the longest Weyl group element; $P_X = B \times B$ and so $L_X = A \times A$, with action on $X^\circ = BwB$ having kernel $\{(a_1, a_2) | a_1 w a_2 = w\}$, i.e. the image of $A \to A \times A$ sending $a \mapsto (a, wa^{-1}w^{-1})$. Thus we have the embedding $\check{A}_X = \check{A} \hookrightarrow \check{A}_G = \check{A} \times \check{A}$ via this same map above, whose extension to $\check{H} \hookrightarrow \check{G} = \check{H} \times \check{H}$ is exactly the identity on the first factor and the Chevalley involution on the second.

In the case $X = G\backslash G = *$, the stabilizer of the orbit is the full group $P_X = L_X = G$; since the torus quotient A_X of P_X is supposed to act faithfully, A_X must be trivial. Looking at our construction above, $k(X) = k(*) = F$ is a one-dimensional trivial G-representation, so the image of $f \mapsto \chi_f$ is trivial; thus $\Xi = \{0\}$ is trivial and so so must $\check{G}_X$ be.

There is an alternative construction of $\check{G}_X$ via the Tannakian formalism due to Gaitsgory–Nadler; the details seem quite difficult but it may be useful as heuristic, although the equivalence of the two constructions is highly nontrivial. If X is a spherical variety, they construct a space Z which morally can be thought of as an avatar of the loop space of X, modified to be finite-dimensional and algebraic (specifically it is an ind-stack). As P_X is the stabilizer of X°, we can write $X^\circ = G/P_X$ so that $G \to X^\circ$ is a P_X-torsor; as X° is an open dense orbit, this means that there is a "generic P_X-torsor" on X. Thus for every loop in X we get a P_X-torsor on the loop by restriction, so for a suitable base curve we get a map $Z \to \mathrm{Bun}_G$; so modifications, which act by Hecke correspondences on Bun_G, act similarly on Z. (Very unsure about all this, it's my attempt to read Gaitsgory–Nadler while sleep-deprived.) One can use this to give an action of sheaves on the affine Grassmannian (via intersection cohomology of a certain subspace of Z), i.e. a functor from Hecke-equivariant perverse sheaves on Gr_G to Hecke-equivariant perverse sheaves on Z; we define a category Q to be the full subcategory of Hecke-equivariant perverse sheaves on Z whose objects are isomorphic to subquotients of the image of this functor.

One can show that this is a tensor category (related to the fusion product, and factorization structures) equivalent to $\mathrm{Rep}(\check{G}_X)$ for some group $\check{G}_X \subset \check{G}$, which turns out to be the same group we constructed. This is reminiscent of the Tannakian construction of $\check{G}$ via geometric Satake; I don't trust my intuition with these objects enough to try and work out the group case, but it's left as an exercise for the interested reader.