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<u>prop</u> If $f(x), g(x)$ are relatively prime, then

$$F[x]/(f(x)g(x)) \cong F[x]/(f(x)) \times F[x]/(g(x))$$

proof: We always have a map $\pi : F[x]/(fg) \to F[x]/(f) \times F[x]/(g)$

which takes $h \mapsto (h \bmod f,\ h \bmod g)$

well-defined because $h + jfg \mapsto (h+0, h+0)$

because any multiple of fg is 0 mod f and mod g.

So we need to see this π is injective and surjective.

Because f, g are relatively prime, we can find a, b so

$$1 = a(x)f(x) + b(x)g(x)$$

$$\begin{aligned} \pi(1) &= \pi(af) + \pi(bg) = (1,1) \\ &= (0, \sim) + (\sim, 0) \quad \left(\begin{matrix}\text{identity maps}\\ \text{to identity}\end{matrix}\right) \\ &= (0,1) + (1,0) \quad \text{because sum is } (1,1) \end{aligned}$$

~~So~~ We can use this formula to find a preimage for any elem of product:

$$(c,d) = c(1,0) + d(0,1)$$

$$d(x)a(x)f(x) + c(x)b(x)g(x) \xmapsto{\pi} d(0,1) + c(1,0)$$

and π is surjective.

Injective: $h \xmapsto{\pi} 0$ means h divisible by f, g rel prime

$\Rightarrow h$ div by $fg \Rightarrow h = 0$ in $F[x]/fg$ //

Linear algebra

$n \times n$ matrix A satisfies its characteristic polynomial $f(x) = \det|A - xI|$: $f(A) = 0$ as matrices

This may not be the minimal polynomial, with this property

ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $f(x) = x^2 - 4x + 3$

$$f(A) = A^2 - 4A + 3I$$
$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

$A = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$ $f(x) = (1-x)(2-x)^2$

$$f(A) = (I-A)(2I-A)^2$$
$$= \begin{bmatrix} 0 & & \\ & -1 & \\ & & -1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

but let $g(x) = (x-1)(x-2)$ divides $f(x)$

$g(A) = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} -1 & & \\ & 0 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ is minimal poly for A.

Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices over $\mathbb{R}$. M is a noncommutative ring with unity I, identity matrix. Let $\mathbb{R}[x]$ be the commutative ring w/ 1 of polynomials with coefs in x. Let $A \in M$ be an $n \times n$ matrix with characteristic polynomial $f(x)$, so $f(A) = 0$.

We have map

$$\boxed{\begin{aligned} \varphi : \mathbb{R}[x] &\longrightarrow M_n(\mathbb{R}) \\ x &\longmapsto A \\ 1 &\longmapsto I \end{aligned}}$$

One way to understand A is to understand this map.

$$\boxed{\varphi(f(x)) = f(A) = 0}$$

(understand why this works!)

e.g. suppose $f(x) = x^2 - 3x + 4$

$$= x \cdot x - 3 \cdot x + 4$$

so $$\varphi(f(x)) = \varphi(x)\varphi(x) - \varphi(3)\varphi(x) + \varphi(4)$$

$$= A\,A - 3I\ A + 4I$$

$$= f(A)$$

Polynomials record ways to add and multiply.

Homomorphisms preserve + and *

We can + and * before or after applying φ

$\Rightarrow$ we can form the polynomial f before or after φ

$$\boxed{\varphi(f(x)) = f(\varphi(x)) = f(A)}$$

This works because φ is a homomorphism, and f is a polynomial, shorthand for a bunch of +, *.

Because $\varphi(f) = 0$, our map is well-defined on the quotient: Redo as

$$\boxed{\begin{array}{c} \varphi : \mathbb{R}[x]\big/(f(x)) \quad 1 \longrightarrow M_n(\mathbb{R}) \\ \\ 1 \longmapsto I \\ x \longmapsto A \end{array}}$$

Noncommutativity: $M_n(\mathbb{R})$ may be noncommutative, but the image of $\mathbb{R}[x]/(f(x))$, image of a commutative ring, has to be commutative. We can ignore everything now except the image, and forget about noncommutativity.

What gives? Powers of A commute:

$$AA^2 = A(AA) = (AA)A = A^2A$$

by associativity alone. The image of $\mathbb{R}[x]/(f(x))$ in $M_n(\mathbb{R})$ is all polynomial expressions in A, all linear combinations of (finitely many) powers of A, taking $A^0 = I$. These expressions form a commutative ring with unit, even though the bigger ring $M_n(\mathbb{R})$ won't be commutative

(This is like working with an abelian subgroup of a nonabelian group.)

Injectivity $\varphi : \mathbb{R}[x]/(f(x)) \longrightarrow M_n(\mathbb{R})$

φ is only injective if $f(x)$ is the <u>minimal</u> polynomial for A. If g strictly dividing f is instead the minimal poly for A, then the ideal (g) is the kernel of φ. (Often we can ignore this issue.)

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We put this together:

If A has char poly $f(x) = g(x)h(x)$ where g, h are relatively prime, then

$$\boxed{\varphi : \mathbb{R}[x]/(gh) \cong \mathbb{R}[x]/(g) \times \mathbb{R}[x]/(h) \longrightarrow M_n(\mathbb{R})}$$

Now we're actually using some ring theory to better understand A. If the factors of the direct product $\mathbb{R}[x]/(g)$, $\mathbb{R}[x]/(h)$ are easier to understand than $\mathbb{R}[x]/(gh)$, then we gain leverage by thinking this way.

Motivating problem: $\boxed{\text{Find } A^n \text{ for some large } n.}$

(This is closely related to the practical problem of finding e^{At}, used to solve systems of differential equations w/ cef matrix A, but has less "overhead" to understand.)

example: find A^{12} for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

direct: multiply out $AAAA.AAAA.AAAA$

better: find x^{12} mod $x^2 - 4x + 3$, map using $x = A$.

repeated squaring trick.

12 in base 2: $8 + 4 = 1100$

so $x^{12} = x^8 x^4$ as product of repeated squares to keep down the size of the polynomials, reduce mod f as we go:

$$x = x$$

$$x^2 = 4x-3 \quad \text{mod } x^2-4x+3$$

$$x^4 = (4x-3)^2$$

$$= 16x^2-24x+9$$

$$= 16(4x-3)-24x+9$$

$$= 40x-39 \quad \text{mod } x^2-4x+3$$

(check: plugging in x=1 should always give 1)

$$x^8 = (40x-39)^2 = (40(x-1)+1)^2$$

$$= \cancel{1600x^2}\ 1600(x-1)^2+80(x-1)+1$$

$$= 1600x^2-3120x+\cancel{1681}\ 1521$$

$$= 1600(4x-3)-3120x+\cancel{1681}\ 1521$$

$$= 3280x-3279 \quad \text{mod } x^2-4x+3$$

so $x^{12} = x^8x^4 = (3280x-3279)(40x-39)$

$$= (3280(x-1)+1)(40(x-1)+1)$$

$$= 3280\cdot 40\,(x-1)^2 + (3280+40)(x-1) + 1$$

$$= 3280\cdot 40\,(\underbrace{4x-3-2x+1}_{2x-2}) + \quad '' \quad ` \quad ' \quad ''$$

$$= (2\cdot 3280\cdot 40 + 3280+40)(x-1)+1$$

so $\boxed{A^{12} = (2\cdot 3280\cdot 40+3280+40)(A-I)+I}$

that was relatively painful, even without simplifying, because we didn't use structure of the ring.

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much easier to do these calculations in each factor, then combine.

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first, let's at least *guess* the answer, so we'll know it when we see it.

$$A^0 = \begin{bmatrix}1&0\\0&1\end{bmatrix} \quad A^1 = \begin{bmatrix}2&1\\1&2\end{bmatrix} \quad A^2 = \begin{bmatrix}5&4\\4&5\end{bmatrix} \quad A^3 = \begin{bmatrix}14&13\\13&14\end{bmatrix}$$

A^n:

n	diag	1^n	3^n	$\frac{1}{2}(1^n+3^n)$	
0	1	1	1	1	
1	2	1	3	2	
2	5	1	9	5	
3	14	1	27	14	
4	41	1	81	41	✓

so $A^n = \begin{bmatrix}\frac{1}{2}(3^n+1) & \frac{1}{2}(3^n-1)\\ \frac{1}{2}(3^n-1) & \frac{1}{2}(3^n+1)\end{bmatrix}$ is our guess.

Nicer way to write this!

$$\boxed{A^n = 1\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\Big/2 + 3^n\begin{bmatrix}1&1\\1&1\end{bmatrix}\Big/2}$$

in particular,

$$I = A^0 = \begin{bmatrix}1&-1\\-1&1\end{bmatrix}\Big/2 + \begin{bmatrix}1&1\\1&1\end{bmatrix}\Big/2$$

$$A = A^1 = \begin{bmatrix}1&-1\\-1&1\end{bmatrix}\Big/2 + 3\begin{bmatrix}1&1\\1&1\end{bmatrix}\Big/2$$

Back to direct product:

$$\mathbb{R}[x]/(gh) = \mathbb{R}[x]/(g) \times \mathbb{R}[x]/(h)$$

$$\mathbb{R}[x]/(x^2-4x+3) = \mathbb{R}[x]/(x-1) \times \mathbb{R}[x]/(x-3)$$

What does x^n look like in $\mathbb{R}[x]/(x-1)$?

$\iff$ What does x^n look like when $x=1$? 1, always

What does x^n look like in $\mathbb{R}[x]/(x-3)$?

$\iff$ What does x^n look like when $x=3$? 3^n

---

we want to glue these answers together to get an answer in $\mathbb{R}[x]/(gh)$, then use $x \mapsto A$ to map this answer to $M_n(\mathbb{R})$, getting a matrix.

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method: we need to find $a(x)$, $b(x)$ so

$$1 = a(x)g(x) + b(x)h(x)$$

using the fact that g, h are relatively prime.

One could use extended euclidean algorithm, or just solve.

Same problem as partial fractions:

$$\frac{1}{g(x)h(x)} = \frac{a(x)}{h(x)} + \frac{b(x)}{g(x)}$$

Linear algebra: From partial fractions, we expect $a(x)=a$, $b(x)=b$ to be constants.

$$1 = a(x-1) + b(x-3)$$

collect terms, equate coeffs:

$$\begin{aligned} a+b &= 0 \\ -a-3b &= 1 \end{aligned}$$

solution: $1 = +\frac{1}{2}(x-1) - \frac{1}{2}(x-3)$ $\left(\begin{aligned} a&=\tfrac{1}{2}, \\ b&=-\tfrac{1}{2} \end{aligned} \right)$

Euclidean algorithm: Practice on $1 = a\cdot 5 + b\cdot 7$

Have basis for plane in $\mathbb{R}^3$ $\{ z = ~~a\cdot 5+b\cdot 7~~ \; a\cdot 5+b\cdot 7 \}$

given by ~~[illegible]~~

$$\begin{aligned} (a,b,z) &= (1,0,5) && \text{from } 5 = 1\cdot 5 \\ &= (0,1,7) && \phantom{\text{from }} 7 = 1\cdot 7 \end{aligned}$$

so all we have to do is find a vector in this plane of the form $(a,b,1)$ and we've found our a,b.

Use euclidean algorithm to reduce 7, 5 to 1, follow along in first two coords to find a,b.

$$\begin{aligned} 7 - 1\cdot 5 &= 2 \qquad & (0,1,7) - 1(1,0,5) &= (-1,1,2) \\ 5 - 2\cdot 2 &= 1 \qquad & (1,0,5) - 2(-1,1,2) &= (3,-2,1) \end{aligned}$$

and sure enough,

$$3\cdot 5 + -2\cdot 7 = 15 - 14 = 1 \;\; ✓$$

Now for real, $\begin{array}{c}(1, 0, x-1)\\(0, 1, x-3)\\\hline\end{array}$ in plane (a,b,z) $\quad a(x-1)+b(x-3)=z$

1st−2nd: $(1,-1,2)$

... ½ : $(\tfrac{1}{2}, -\tfrac{1}{2}, 1)$

$$1 = \tfrac{1}{2}(x-1) - \tfrac{1}{2}(x-3)$$

---

as before $\xmapsto{\pi}$ mod $x-1$, mod $x-3$

$$\begin{array}{rcll}
\tfrac{1}{2}(x-1) & \longmapsto & 0, & \sim \\
-\tfrac{1}{2}(x-3) & \longmapsto & \sim, & 0 \\
\text{but sum} \quad 1 & \longmapsto & 1, & 1
\end{array}$$

so
$$\tfrac{1}{2}(x-1) = (0,1)$$
$$-\tfrac{1}{2}(x-3) = (1,0)$$

and we know how to match up answers in $\mathbb{R}[x]/gh$ with $\mathbb{R}[x]/(x-1) \times \mathbb{R}[x]/(x-3)$

$$x^n = (1,0) + 3^n(0,1)$$
$$x^n = -\tfrac{1}{2}(x-3) + 3^n\left[\tfrac{1}{2}(x-1)\right]$$

mod $(x-1)(x-3)$

now send $x \mapsto A$:

$$\boxed{A^n = \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}/2 + 3^n\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}/2}$$

as we guessed !!

Same approach works for e^{At}:

In $\mathbb{R}[x]/_{(x-1)}$, x is like 1, so e^{xt} is like e^{t}

" $\mathbb{R}[x]/_{(x-3)}$, x " " 3, " e^{xt} " " e^{3t}

chasing answer back to matrices, via the ring $\mathbb{R}[x]/_{(x-1)(x-3)}$,

we get

$$\boxed{e^{At} = e^{t}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/_{2} + e^{3t}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/_{2}}$$

The 3×3 examples are already in Jordan canonical form, to keep the calculations simple. The beauty of the method is that we evaluate the same polynomials to get the right answer, even if A is in the wrong coord system, not diagonalized or Jordan form:

$A = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix}$ has characteristic polynomial $(x-2)(x-3)^2$ (we fixed the sign) but minimal polynomial $(x-2)(x-3)$.

Suppose we knew this. We'd work with

$$\mathbb{R}[x]/_{(x-2)(x-3)} \cong \mathbb{R}[x]/_{(x-2)} \times \mathbb{R}[x]/_{(x-3)} \longrightarrow M_3(\mathbb{R})$$

and send $x \longmapsto A$ to compute functions of ~~x~~ A.

Euclidean algorithm:
$$\begin{array}{c} (1, 0, x-2) \\ (0, +1, x-3) \\ \hline (1, -1, 1) \end{array}$$

so $1 = (x-2) - (x-3)$

$$(1,1) = (0,1) + (1,0) \quad \text{in} \quad \mathbb{R}[x]/(x-2) \times \mathbb{R}[x]/(x-3)$$

e^{xt} looks like (e^{2t}, e^{3t})

$$e^{xt} = (e^{2t}, e^{3t}) = e^{2t}(1,0) + e^{3t}(0,1)$$
$$= e^{2t}(-(x-3)) + e^{3t}(x-2)$$

$x \mapsto A$:

$$e^{At} = e^{2t}(-(A-3I)) + e^{3t}(A-2I)$$

$A = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix}$

$$= e^{2t}\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{2t} & & \\ & e^{3t} & \\ & & e^{3t} \end{bmatrix}$$ as we expected,

exponentiating a diagonal matrix ~~diagonalizes A~~ exponentiates the entries.

Had A been messy nondiagonal matrix, above formulas would have saved us a lot of work.

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What if we had been working with $(x-2)(x-3)^2$?

$(x-2)$, $(x-3)^2$ are relatively prime, so as before,

$$\boxed{\mathbb{R}[x]/_{(x-2)(x-3)^2} \cong \mathbb{R}[x]/_{(x-2)} \times \mathbb{R}[x]/_{(x-3)^2} \longrightarrow M_3(\mathbb{R})}$$

makes sense for working with any A so $(A-2)(A-3)^2=0$, whether or not $(x-2)(x-3)^2$ is the <u>minimal</u> polynomial for A. The method still works, it's just unnecessary effort when we're not using the minimal polynomial.

$\mathbb{R}[x]/_{(x-2)}$ is as before. $\mathbb{R}[x]/_{(x-3)^2}$ is harder.

$\mathbb{R}[x]/_{(x-3)^2}$ has as vector space basis over $\mathbb{R}$ $\{1, x\}$.

we can reduce any higher power of x mod $(x-3)^2$ using $(x-3)^2=0 \Leftrightarrow x^2-6x+9=0 \Leftrightarrow x^2=6x-9$.

So what is e^{xt} in $\mathbb{R}[x]/_{(x-3)^2}$?

$$x = (x-3)+3 \quad \text{so} \quad e^{xt} = e^{(x-3)t}e^{3t}$$

$$e^{(x-3)t} = 1 + (x-3)t + \underbrace{\tfrac{1}{2}(x-3)^2t^2 + \tfrac{1}{6}(x-3)^3t^3 + }_{0 \text{ mod } (x-3)^2}$$

so $$\boxed{e^{xt} = [1+(x-3)t]\,e^{3t} \quad \text{in } \mathbb{R}[x]/_{(x-3)^2}}$$

putting this together,

$$e^{xt} = \left(e^{2t}, [1+(x-3)t]e^{3t}\right) \quad \text{in} \quad \frac{\mathbb{R}[x]}{(x-2)} \times \frac{\mathbb{R}[x]}{(x-3)^2}$$

$$= e^{2t}(1,0) + [1+(x-3)t]e^{3t}(0,1)$$

aha, we haven't yet done euclidean algorithm:

$$\begin{array}{l} (1,\ 0,\ x-2) \\ (0,\ 1,\ x^2-6x+9) \\ \hline (-x,\ 1,\ -4x+9) \\ (4-x,\ 1,\ 1) \end{array} \Rightarrow \underset{(0,1)}{\underset{\shortparallel}{(4-x)(x-2)}} + \underset{(1,0)}{\underset{\shortparallel}{(x-3)^2}} = \underset{(1,1)}{\underset{\shortparallel}{1}} \quad \checkmark$$

$$e^{xt} = e^{2t}(x-3)^2 + [1+(x-3)t]e^{3t}(4-x)(x-2)$$

$$= e^{2t}\left[(x-3)^2\right] + e^{3t}\left[(4-x)(x-2)\right] + \underbrace{te^{3t}\left[(4-x)(x-2)\cdot(x-3)\right]}$$

before even plugging in $x=A$, we can see that the te^{3t} term gives 0 whenever $(A-2)(A-3)=0$.

So when $(x-2)(x-3)$ is ~~a~~ the minimal polynomial, we did extra work by instead using $(x-2)(x-3)^2$.

We should still get the right answer:

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$$\boxed{e^{xt} = e^{2t}\left[(x-3)^2\right] + e^{3t}\left[(4-x)(x-2)\right] + te^{3t}\left[(4-x)(x-2)(x-3)\right]}$$

(mod $(x-2)(x-3)^2$)

now $x \mapsto A = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix}$

$$e^{At} = e^{2t}\left[\begin{bmatrix} -1 & & \\ & 0 & \\ & & 0 \end{bmatrix}^2\right] + e^{3t}\left[\begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right] + te^{3t}\left[\begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}\underbrace{\begin{bmatrix} -1 & & \\ & 0 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{0}\right]$$

$$= e^{2t}\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \text{ as before.}$$

What about an example that <u>needs</u> $(x-2)(x-3)^2$?
That is the characteristic polynomial of

$$A = \begin{bmatrix} 2 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$$

which is in Jordan form but cannot be diagonalized.

now $x \mapsto A$:

$$e^{At} = e^{2t}\left(\begin{bmatrix} -1 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix}^2\right) + \cancel{3}e^{3t}\left(\overbrace{\begin{bmatrix} 2 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix}\begin{bmatrix} 0 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}}^{\text{same}}\right) + te^{3t}\left[\overbrace{\begin{bmatrix} 2 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix}\begin{bmatrix} 0 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}}\begin{bmatrix} -1 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix}\right]$$

$$= e^{2t}\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} + te^{3t}\begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

gives familiar $e^{At} = \begin{bmatrix} e^{2t} & & \\ & e^{3t} & te^{3t} \\ & & e^{3t} \end{bmatrix}$

and is fast method when A isn't in such nice coordinates.

Summary of what we've done:

If $f(A)=0$ for a matrix A and polynomial $f(x)$ (such as the characteristic poly $\det(xI-A)$) then we have a ring homomorphism

$$\varphi : \mathbb{R}[x]/(f) \longrightarrow M_n(\mathbb{R})$$

given by

$$\begin{aligned} 1 &\longmapsto I \\ x &\longmapsto A \end{aligned}$$

If now $k(x)$ is a polynomial in x, we can compute $k(A)$ by first computing $k(x) \bmod f(x)$, i.e. computing in the ring $\mathbb{R}[x]/(f)$ then applying φ.

(Short version: If $f(A)=0$ we can compute any $k(x)$ for $x=A$ by first working mod $f(x)$.)

If now $f(x)=g(x)h(x)$ for g, h relatively prime, then for some $a(x)$, $b(x)$,

$$\boxed{1 = a(x)g(x) + b(x)h(x)}$$

Then

$$\mathbb{R}[x]/(g) \times \mathbb{R}[x]/(h) \longrightarrow \mathbb{R}[x]/(f)$$

is an explicit isomorphism, using a and b:

we know what the isomorphism looks like the other way,

$$\pi : \mathbb{R}[x]/(f) \longrightarrow \mathbb{R}[x]/(g) \times \mathbb{R}[x]/(h)$$

$$\begin{aligned} & x \bmod f && \longmapsto && x \bmod (g,h) \\ & 1 && \longmapsto && (1,1) \\ = \; & ag && \longmapsto && (0,1) \\ + \; & bh && \longmapsto && \text{[illegible]} \; (1,0) \end{aligned}$$

$$\begin{aligned} \text{so} \quad ag &\cong (0,1) \\ bh &\cong (1,0) \end{aligned}$$

Now we can map any element (c,d) back

$$\mathbb{R}[x]/(g) \times \mathbb{R}[x]/(h) \xrightarrow{\sim} \mathbb{R}[x]/(f)$$

$$\begin{aligned} (c,d) &= c(1,0) + d(0,1) \\ &\longmapsto c(bh) + d(ag) \end{aligned}$$

Putting this together, we can compute $k(x)$ separately mod g and mod h, map to mod f, and plug in $x=A$:

$$\boxed{\begin{aligned} & \mathbb{R}[x]/(g) \times \mathbb{R}[x]/(h) \xrightarrow{\sim} \mathbb{R}[x]/(f) \longrightarrow M_n(\mathbb{R}) \\ & (c,d) \longmapsto cbh + dag \quad , \qquad x \mapsto A \end{aligned}}$$

This means we can reduce the computation of $k(A)$ for a function $k(x)$ to computing $k(x) \bmod (x-\lambda)^m$ for each eigenvalue λ of A of multiplicity m. (We may need to move from $\mathbb{R}$ to $\mathbb{C}$ to find all roots λ of $f(x)$.)

Suppose that $k(x)$ is a continuous function $k: \mathbb{R} \to \mathbb{R}$, sufficiently differentiable at $x = \lambda$. Then we can expand $k(x)$ as a Taylor series in $(x-\lambda)$:

$$k(x) = k(\lambda) + k'(\lambda)(x-\lambda) + \frac{k''(\lambda)}{2}(x-\lambda)^2 + \dots$$

In the ring $\mathbb{R}[x]/(x-\lambda)^m$, all terms of deg $\geq m$ in this series vanish, and $k(x)$ becomes a polynomial. Doing this for each root λ of $f(x)$, we can use the preceding theory to compute $k(A)$.

(There are convergence issues to tackle in an analysis course, but if there is an answer to $k(A)$, this finds it.)