Practice Final Exam

Modern Algebra II, Dave Bayer, December 2010

Name: ______________________________

[1] (4 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	[6] (6 pts)	[7] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[**1**] The polynomial

$$g(a, b) = (a - b)^4$$

is symmetric in a and b. Express g as a polynomial in the elementary symmetric functions

$$s_1 = a + b, \quad s_2 = ab.$$

[**2**] The polynomial

$$g(a, b, c) = a^3 + b^3 + c^3$$

is symmetric in a, b, and c. Express g as a polynomial in the elementary symmetric functions

$$s_1 = a + b + c, \quad s_2 = ab + ac + bc, \quad s_3 = abc.$$

[**3**] The polynomial

$$g(a, b, c) = (a - b)^2(a - c)^2(b - c)^2$$

is symmetric in a, b, and c. Suppose that

$$s_1 = a + b + c = 0.$$

Express g as a polynomial in the remaining elementary symmetric functions

$$s_2 = ab + ac + bc, \quad s_3 = abc.$$

[**4**] What is the irreducible polynomial for $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$?

[**5**] Let $f(x) = x^3 - 12$. What is the degree of the splitting field K of f over $\mathbb{Q}$?
What is the Galois group $G = G(K/\mathbb{Q})$ of f?
List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

[**6**] Which of the following cubic polynomials have A_3 for their Galois group? Which have S_3 for their Galois group?

$$x^3 - 21x + 7, \qquad x^3 - 3x^2 + 1, \qquad x^3 + x^2 + x + 1$$

[7] Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^2 - 2x + 3$. Find an element $a \in \mathbb{Q}$ such that $K = \mathbb{Q}(\sqrt{a})$.

[8] Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 + px + q$, where $p, q \in \mathbb{Q}$. When is the degree $[K : \mathbb{Q}] = 3$? When is the degree $[K : \mathbb{Q}] = 6$? Give an example of a polynomial for each case.

[9] Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^5 - 81x + 3$. What is the Galois group $G(K/\mathbb{Q})$?

[10] Let F be the splitting field of the polynomial $x^p - 1$ over $\mathbb{Q}$, where p is a prime. What is the Galois group $G(F/\mathbb{Q})$?

[11] Let F be the splitting field of the polynomial $x^n - 1$ over $\mathbb{Q}$, where n is a positive integer. What is the Galois group $G(F/\mathbb{Q})$?

[12] Give an example of a degree two polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is C_2.

[13] Give an example of a degree three polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is A_3.

[14] Give an example of a degree three polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is S_3.

[15] Give an example of a degree four polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $C_2 \times C_2$.

[16] Give an example of a degree four polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is C_4.

[17] Give an example of a degree five polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is S_5.

[18] Give an example of a degree six polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is C_6.

Proofs

[19] Let $K = F(\alpha, \beta)$ be a finite extension of a field F of characteristic zero. Prove that there is an element $\gamma \in K$ such that $K = F(\gamma)$.

[20] Let F be a subfield of $\mathbb{C}$ that contains all roots of the polynomial $x^p - 1$, where p is a prime. Let K/F be a Galois extension of degree p. Prove that $K = F(\sqrt[p]{b})$ for some $b \in F$.