Practice Problems for Final Exam

Modern Algebra, Dave Bayer, May 3, 1999

Our final will be held on Wednesday, May 12, 1:10pm – 4:00pm, in our regular classroom. It will consist of 8 questions worth 40 points in all. Two questions will be review from previous exams, and the remaining six questions will test material since the last exam.

The following two problems constitute the review topics from previous exams.

[1] (compare with midterm 1, problem 5) Let $\mathbf{X} \subset \mathbb{R}^2$ be a finite set of points. Define $I \subset \mathbb{R}[x, y]$ to be the set of all polynomials $f(x, y)$ that vanish on every point of $\mathbf{X}$. That is,

$$I \quad = \quad \{ \; f(x,y) \in \mathbb{R}[x,y] \;\mid\; f(a,b) = 0 \;\text{ for every point }\; (a,b) \in X \;\}.$$

Prove that I is an ideal.

[2] (midterm 2, problem 2) What is the minimal polynomial of $\alpha = \sqrt{-1} + \sqrt{2}$ over $\mathbf{Q}$? (Note that we now have another way to compute this; see Artin, p. 554, discussion after proof of Proposition 14.4.4.)

The following problems are taken from last year's review questions for the final.

[3] Let $f(x, y, z) = x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2$. Express $f(x)$ as a polynomial $g(\sigma_1, \sigma_2, \sigma_3)$ where σ_1, σ_2, σ_3 are the elementary symmetric functions

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + xz + yz, \quad \sigma_3 = xyz.$$

[4] Let $f(x) = x^2 + ax + b$ have roots α_1 and α_2, where

$$\alpha_1 + \alpha_2 = c, \qquad \alpha_1^2 + \alpha_2^2 = d.$$

Express a and b in terms of c and d.

Recall that the discriminant of $f(x) = x^2 + bx + c$ is $D = b^2 - 4c$, and that the discriminant of $f(x) = x^3 + px + q$ is $D = -4p^3 - 27q^2$.

[5] Let $f(x) = x^3 - 2$.

(a) What is the degree of the splitting field K of f over $\mathbf{Q}$?

(b) What is the Galois group $G = G(K/\mathbf{Q})$ of f?

(c) List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

[6] Repeat for $f(x) = x^3 - 3x + 1$.

[7] Repeat for $f(x) = x^4 - 3x^2 + 2$.

[8] Repeat for $f(x) = x^4 - 5x^2 + 6$.

[9] Prove the primitive element theorem (14.4.1, p. 552): Let K be a finite extension of a field F of characteristic zero. There is an element $\gamma \in K$ such that $K = F(\gamma)$.

[10] Prove the following theorem about Kummer extensions (14.7.4, p. 566): Let F be a subfield of $\mathbf{C}$ which contains the pth root of unity ζ for a prime p, and let K/F be a Galois extension of degree p. Then K is obtained by adjoining a pth root to F.

The following problems are taken from last year's final.

[11] Prove that $\alpha = e^{2\pi i/11} + 3$ is not constructible.

[12] What is the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$?

[13] Let $f(x, y, z) = x^3 + y^3 + z^3$. Express $f(x)$ as a polynomial $g(\sigma_1, \sigma_2, \sigma_3)$ where σ_1, σ_2, σ_3 are the elementary symmetric functions

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + xz + yz, \quad \sigma_3 = xyz.$$

Recall that the discriminant of $f(x) = x^2 + bx + c$ is $D = b^2 - 4c$, and that the discriminant of $f(x) = x^3 + px + q$ is $D = -4p^3 - 27q^2$.

[14] Let $f(x) = x^3 - 12$.
(a) What is the degree of the splitting field K of f over $\mathbf{Q}$?
(b) What is the Galois group $G = G(K/\mathbf{Q})$ of f?
(c) List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

[15] Prove the primitive element theorem (14.4.1, p. 552): Let K be a finite extension of a field F of characteristic zero. There is an element $\gamma \in K$ such that $K = F(\gamma)$.