Practice Problems

Combinatorics, Dave Bayer, September 22, 2011

[1] How many permutations of n letters have exactly d fixed points? Give the best answer you can, using related problems.

[2] How many ways can one make change, using 1 cent, 2 cent, and 3 cent coins? What if one is not allowed to use all three kinds of coins at once? What if the number of 1 cent and 2 cent coins are restricted to differ by no more than two coins? Express your answers as generating functions.

[3] How many monomials are there of degree d in the four variables w, x, y, and z? Of these monomials, how many involve at most two variables? How many involve at least two variables? Express your answers as formulas, and as generating functions.

[4] How many different Venn diagrams are possible in the plane for three properties, if each pair of regions intersect in at most one connected region? For each diagram, draw the corresponding partial order of possible combinations of properties. Can you give examples of counting problems corresponding to each of your diagrams? (Two diagrams are the same if they differ only by the names of the properties.)

[5] Consider the recurrence

$$F(n,d) \;=\; F(n-1,d) \;+\; F(n,d-n)$$

where $F(n,d) = 0$ if either n or d is negative. Let $F(0,0) = 1$, and $F(0,d) = 0$ for $d > 0$. Define

$$G_n(t) \;=\; \sum_{d=0}^{\infty} F(n,d)\, t^d$$

Solve for the generating functions $G_n(t)$. Can you think of a natural problem where this recurrence arises?