Exam 1

Combinatorics, Dave Bayer, October 4, 2011

Name: Answers

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] How many integers in the sequence

$$1,\ 2,\ 3,\ 4,\ \ldots\ 718,\ 719,\ 720$$

are not divisible by 4, 5, or 6?

720 in all

$- \ 720/4 = 180$ divisible by 4

$- \ 720/5 = 144$ " " 5

$- \ 720/6 = 120$ " " 6

$+ \ 720/20 = 36$ 4 and 5

$+ \ 720/30 = 24$ 5 and 6

$+ \ 720/12 = 60$ 4 and 6 (lcm 12)

$- \ 720/60 = 12$ 4, 5, and 6 (lcm 60)

$$720 - (\underbrace{180+144+120}_{444}) + (\underbrace{36+24+60}_{120}) - 12$$

$$720 - 336 = \boxed{384}$$

[2] Count the number of ways of making change for n cents using pennies and nickels. Give a table of small values, and a generating function. For example, there are two ways of making change for 8 cents, namely

$$1+1+1+1+1+1+1+1, \qquad 1+1+1+5.$$

n	# ways
0	1
1	1
2	1
3	1
4	1
5	2
6	2
7	2
8	2
9	2
10	3
11	3
$\vdots$	

Let a be a penny
b " nickel

$$(1+a+a^2+\dots)(1+b+b^2+\dots)$$

$$= \frac{1}{1-a}\,\frac{1}{1-b}$$

setting $a=t$, $b=t^5$,

$$\boxed{G = \frac{1}{1-t}\cdot\frac{1}{1-t^5}}$$

(For example, a^8 and a^3b are the two ways to make change for 8 cents.)

[3] Count the number of words using the letters 2 and 3 whose letters add up to n. Give a table of small values, and a recurrence relation. For example, there are four words that add up to 8, namely

2222, 233, 323, 332.

F_n — start with 2: F_{n-2} "2 $\underbrace{\text{~~~~~}}_{\text{adds to } n-2}$"

don't " ": F_{n-3} "3 $\underbrace{\text{~~~~~}}_{\text{adds to } n-3}$"

n	# ways
0	1
1	0
2	1
3	1
4	1
5	2
6	2
7	3
8	4
9	5
10	7
11	9
⋮	

initial values (n = 0, 1, 2)

$$\boxed{F_n = F_{n-2} + F_{n-3}}$$

[4] The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

only valid for $n \geq 2$

with initial values $F_0 = 0$ and $F_1 = 1$. Solve for the generating function

$$G = \sum_{n=0}^{\infty} F_n t^n$$

$$\sum_{n=2}^{\infty} \left(F_n = F_{n-1} + F_{n-2} \right) t^n$$

$$\sum_{n=2}^{\infty} F_n t^n = \sum_{n=2}^{\infty} F_{n-1} t^n + \sum_{n=2}^{\infty} F_{n-2} t^n$$

$$\sum_{n=2}^{\infty} F_n t^n = t \sum_{n=1}^{\infty} F_n t^n + t^2 \sum_{n=0}^{\infty} F_n t^n$$

$$G - F_0 - F_1 t = t(G - F_0) + t^2 G$$

$$G - t = tG + t^2 G$$

$$G - tG - t^2 G = t$$

$$\boxed{G = t/_{1-t-t^2}}$$

[5] Count the number of ways of making change for n cents using 5, 6, and 7 cent coins. Give your answer as a generating function. Of these, how many ways *do not contain* a way to make change for twelve cents? For example,

$$5+5+5+5+5+5, \qquad 6+6+6+6+6$$

are two of the possible ways to make change for 30 cents, but one of them contains a way to make change for 12 cents.

$a = 5$ cents $\qquad t^5$
$b = 6$ cents $\quad \Rightarrow \quad t^6$
$c = 7$ cents $\qquad t^7$

all ways, $\dfrac{1}{(1-a)(1-b)(1-c)} \Rightarrow \dfrac{1}{(1-t^5)(1-t^6)(1-t^7)}$

There are two ways to make change for 12 cents, ac and b^2. If we subtract both of them, we have to add back in ab^2c:

$$\frac{1-ac-b^2+ab^2c}{(1-a)(1-b)(1-c)} = \frac{(1-ac)(1-b^2)}{(1-a)(1-b)(1-c)}$$

$$\Rightarrow \boxed{\frac{1-2t^{12}+t^{24}}{(1-t^5)(1-t^6)(1-t^7)} = \frac{(1-t^{12})^2}{(1-t^5)(1-t^6)(1-t^7)}}$$

Another way:

- use b at most once
- use a or c but not both

$(1+b)\left[\dfrac{1}{1-a} + \dfrac{1}{1-c} - 1\right]$ gives same answer

[6] Approximately what fraction of all permutations of n letters have no fixed points? One fixed point? Two fixed points? If you keep going for k fixed points for each k, do your answers add up to one?

$\approx 1/e$ have no fixed points

$\binom{n}{1} F_{n-1}/n!$ have one fixed point, where

$F_n = \#$ derangements of n letters

$F_n/n! \approx 1/e$

So $\binom{n}{1} F_{n-1}/n! = F_{n-1}/(n-1)! \approx 1/e$ also have <u>one</u> fixed point.

$\binom{n}{2} F_{n-2}/n! = \frac{1}{2} F_{n-2}/(n-2)! \approx 1/2e$ have <u>two</u> fixed points.

$$1/e + 1/e + 1/2e + 1/6e + \ldots$$

$$= 1/e(1+1+1/2+1/6+\ldots)$$

$$= 1/e \cdot e = 1 \quad \checkmark$$

Yes, the answers add up to one.