Practice Problems

Combinatorics, Dave Bayer, November 10, 2011

[1] Prove *Burnside's lemma*: Let G be a finite group that acts on a finite set X. If X^g denotes the set of elements of X that are fixed by an element $g \in G$, then the number of orbits of this action is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

[2] Demonstrate an example of the use of *Burnside's lemma* to solve a counting problem, taking into account symmetry.

[3] How many ways can we place two markers on a 5 by 5 checkerboard, up to rotational symmetry?

[4] How many different necklaces can be made from six beads, if three colors of beads are available, taking into account symmetry? (A necklace can be flipped and rotated.)

[5] How many ways can a five term product be fully parenthesized? For example, $(ab)(c(de))$ and $a(b(c(de)))$ are two different ways to multiply together $abcde$. Relate this count to trees, and draw the trees.

[6] There are 16 labeled trees on four vertices. Draw them.

[7] A labeled tree on six vertices has the Prüfer sequence {4,4,4,5}. Draw the tree.

[8] A *planar cubic graph* is a graph that can be drawn in the plane, all of whose vertices have degree three. Find examples of planar cubic graphs on 4, 6, 8 and 10 vertices. How many distinct graphs can you find?

[9] Find the smallest *planar cubic graph* that can be separated into two components by deleting two edges.

[10] Show that a *planar cubic graph* must have an even number of vertices.

[11] Show that a *planar graph* must have a vertex of degree five or less.

[12] Prove the *six color theorem* for planar graphs: Every planar graph has a proper vertex coloring using only six colors.

[13] Prove the *five color theorem* for planar graphs: Every planar graph has a proper vertex coloring using only five colors.