Exam 2

Combinatorics, Dave Bayer, November 17, 2011

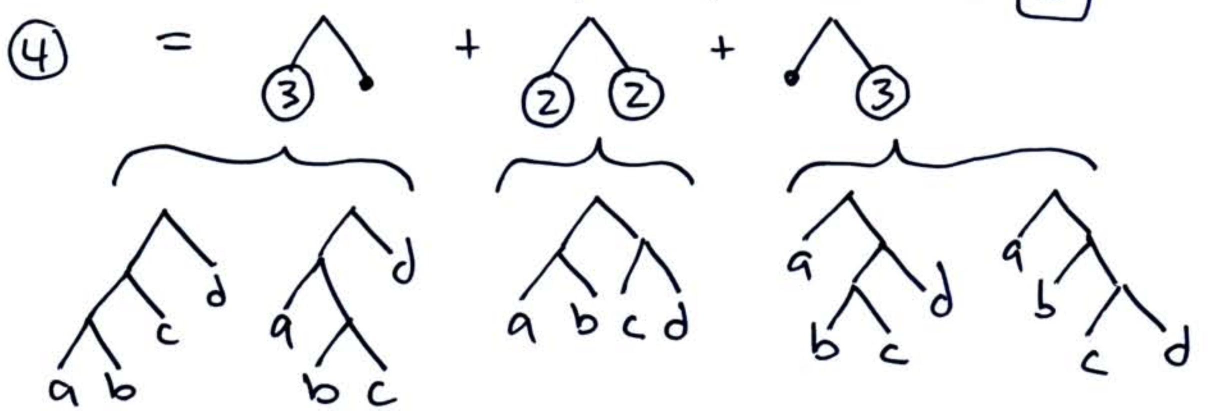
Name: Solutions

[1] (6 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

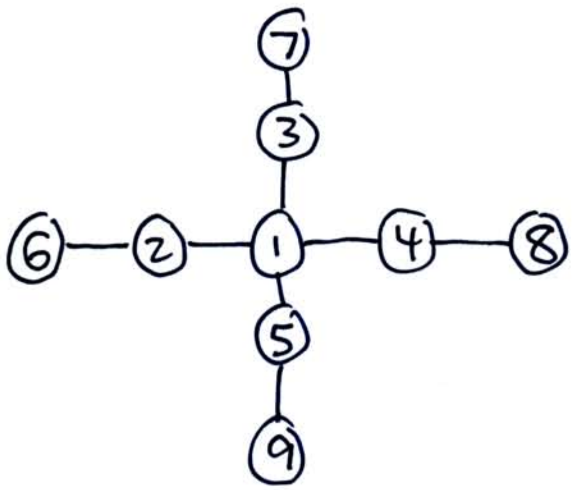
[1] How many ways can a four term product be fully parenthesized? For example, $((ab)c)d$ and $(ab)(cd)$ are two different ways to multiply together $abcd$. Relate this count to binary trees, and draw the trees.

$2 + 1 + 2 = \boxed{5}$

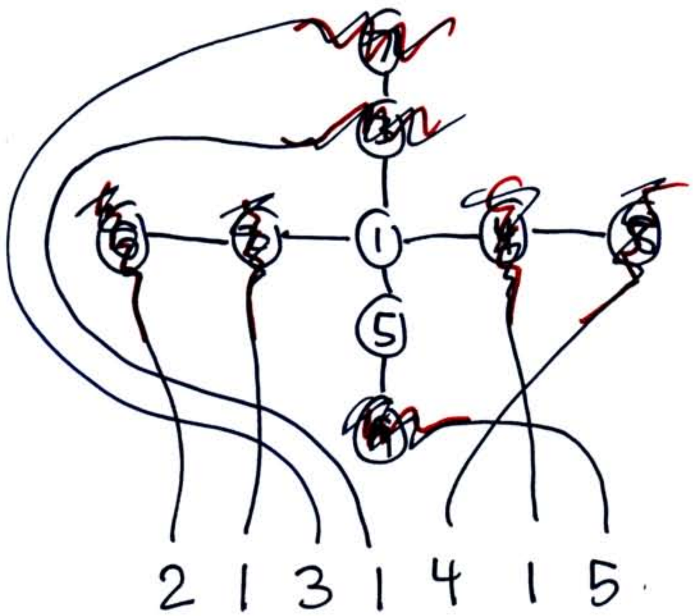


$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$

[2] A labeled tree on nine vertices has the Prüfer sequence 2,1,3,1,4,1,5. Draw the tree.



check:



[3] How many different necklaces can be made from five beads, if three colors of beads are available, taking into account symmetry? (A necklace can be flipped and rotated.)

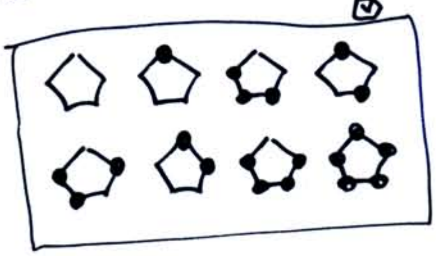
Of these necklaces, how many use all three colors?

$$G = \{1, \underbrace{\text{turns}}_{4}, \underbrace{\text{flips}}_{5}\} \qquad \text{orbits} = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

one color: 1

two colors: $\frac{1}{10}(2^5 + 4\cdot 2^1 + 5\cdot 2^3) = \frac{1}{10}(32+8+40) = 8$

○ 5 choices	1 choice	3 choices
• none forced	4 forced	2 forced
2^5	2^1	2^3

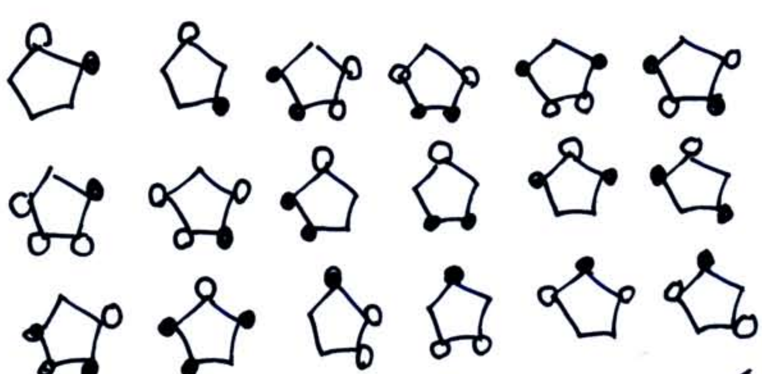


three colors: (at most) $\frac{1}{10}(\underset{243}{3^5} + \underset{12}{4\cdot 3^1} + \underset{135}{5\cdot 3^3}) = \boxed{39}$

all three colors: $39 - 3\cdot 8 + 3\cdot 1 - 0 = \boxed{18}$

(at most 3) − 3(only 2) + 3(only 1) − (none)

inclusion-exclusion counting (!)



who is special

3+1+1 special

2+2+1 special

whew!

(One can also directly use Burnside's lemma. Harder, but it works)

[4] Prove *Burnside's lemma*: Let G be a finite group that acts on a finite set X. If X^g denotes the set of elements of X that are fixed by an element $g \in G$, then the number of orbits of this action is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

$$\sum_{g \in G} |X^g| = |\{(g,x) \mid gx = x\}| = \sum_{x \in X} |G_x|$$

← subgroup that fixes x

$g \in G$  does $gx = x$?

$x \in X$

count by columns instead of rows

If $\mathcal{O}_x \subset X$ is the orbit of $x \in X$, then $|G| = |G_x||\mathcal{O}_x|$

(throw elems of G into bins according to gx, the bins all have same size.)

$$\Rightarrow \quad |G_x| = \frac{|G|}{|\mathcal{O}_x|} \quad \text{so}$$

$$\sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|\mathcal{O}_x|} = |G| \sum_{x \in X} \frac{1}{|\mathcal{O}_x|} = |G| \sum_{\substack{\text{orbits} \\ \mathcal{O}_x}} \left(\sum_{x \in \mathcal{O}_x} \frac{1}{|\mathcal{O}_x|} \right)$$

$$= |G| \sum_{\substack{\text{orbits} \\ \mathcal{O}_x}} 1 = |G| \, (\# \text{ orbits } \mathcal{O}_x)$$

so $$\frac{1}{|G|} \sum |X^g| = \# \text{ orbits } \mathcal{O}_x$$

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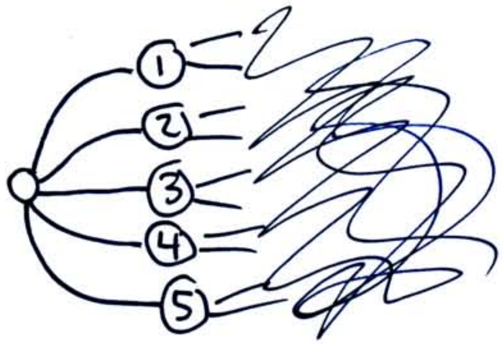
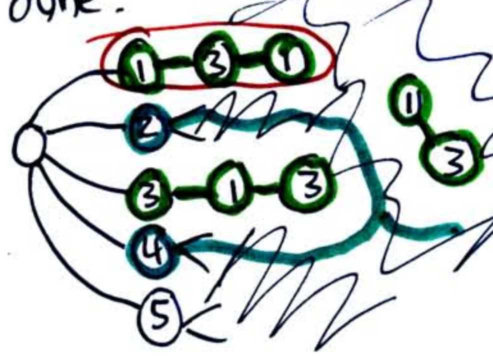
[5] Prove the *five color theorem* for planar graphs: Every planar graph has a proper vertex coloring using only five colors. (You may assume that every planar graph has a vertex of degree five or less.)

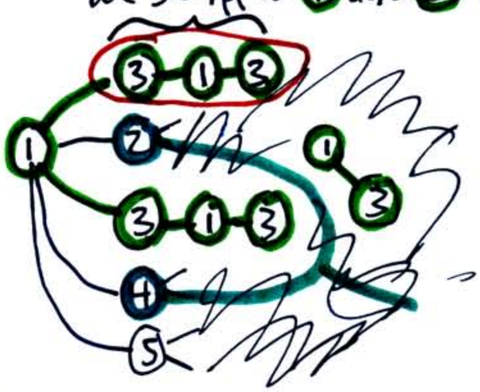


Pick a vertex of degree at most five. Inductively color the rest of the graph using five colors. If the vertex you left out only has four colors as neighors, you're done: color it using the missing color.

Otherwise, the uncolored vertex has degree five, with all five colors as neighbors. Number them clockwise 1, 2, 3, 4, 5, so any path from 1 to 3 has to cross any path from 2 to 4.

Highlight all vertices 1 and 3 green, and the edges between them. If there is no highlighted path between the 1 and 3 neighbors of the last vertex, then just swap 1 and 3 on the connected piece of the 1 neighbor of the last vertex. Now 1 is available, use it to color the last vertex and you are done.





Otherwise, highlight 2 and 4 blue
They can't connect because 1 and 3 are in the way, so you can free 2.

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