Exam 2

Linear Algebra, Dave Bayer, November 6-9, 2020

[1] Find an orthogonal basis for $\mathbb{R}^4$ that includes the vector $(1, 1, 1, 0)$.

① Start with $v_1 = [1\ 1\ 1\ 0]$

② Find a nonzero solution to $[1\ 1\ 1\ 0]\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = [0]$ to find $v_2 \perp v_1$

$$[1\ 1\ 1\ 0]\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = [0] \qquad v_2 = [0\ 0\ 0\ 1]$$

③ Find a nonzero solution to $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find $v_3 \perp v_1, v_2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_3 = [1\ {-1}\ 0\ 0]$$

④ Find a nonzero solution to $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to find $v_4 \perp v_1, v_2, v_3$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_4 = [1\ 1\ {-2}\ 0]$$

answer: (One of many possible answers)

$$\boxed{\begin{aligned} v_1 &= [1\ 1\ 1\ 0] \\ v_2 &= [0\ 0\ 0\ 1] \\ v_3 &= [1\ {-1}\ 0\ 0] \\ v_4 &= [1\ 1\ {-2}\ 0] \end{aligned}}$$

Check that each pair dots to zero $\Rightarrow \perp$

v_1	v_2	v_3
✓		
✓	✓	
✓	✓	✓

(One could also use Gram-Schmidt.)

[2] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Find an orthogonal basis for V with respect to the inner product

$$f(x) = ax^2+bx+c$$
$$g(x) = rx^2+sx+t$$

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$\begin{aligned}\langle f,g\rangle &= \int_0^1 (ax^2+bx+c)(rx^2+sx+t)\,dx \\ &= \int_0^1 \left[(ar)x^4+(as+br)x^3+(at+bs+cr)x^2+(bt+cs)x+(ct)\right]dx \\ &= \tfrac{1}{5}(ar)+\tfrac{1}{4}(as+br)+\tfrac{1}{3}(at+bs+cr)+\tfrac{1}{2}(bt+cs)+(ct) \\ &= \begin{bmatrix} a & b & c\end{bmatrix}\begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1\end{bmatrix}\begin{bmatrix} r \\ s \\ t\end{bmatrix}\end{aligned}$$

1 Start with $v_1 = \begin{bmatrix} 0 & 0 & 1\end{bmatrix}$

2 Find a nonzero solution to to find $v_2 \perp v_1$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1\end{bmatrix}}\begin{bmatrix} r \\ s \\ t\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/2 & 1\end{bmatrix}\begin{bmatrix} r \\ s \\ t\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/2 & 1\end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ -1\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

so $v_2 = \begin{bmatrix} 0 & 2 & -1\end{bmatrix}$

3 Find a nonzero solution to to find $v_3 \perp v_1, v_2$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -1\end{bmatrix}\begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1\end{bmatrix}}\begin{bmatrix} r \\ s \\ t\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/2 & 1 \\ 1/6 & 1/6 & 0\end{bmatrix}\begin{bmatrix} r \\ s \\ t\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/2 & 1 \\ 1/6 & 1/6 & 0\end{bmatrix}\begin{bmatrix} 6 \\ -6 \\ 1\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $v_3 = \begin{bmatrix} 6 & -6 & 1\end{bmatrix}$

answer:

$$\boxed{\begin{aligned} v_1 &= 1 \\ v_2 &= 2x-1 \\ v_3 &= 6x^2-6x+1\end{aligned}}$$

(One of many possible answers)

check:

$$\int_0^1 v_1v_2\,dx = \int_0^1 (2x-1)\,dx = 2/2 - 1 = 0 \checkmark$$
$$\int_0^1 v_1v_3\,dx = \int_0^1 (6x^2-6x+1)\,dx = 6/3 - 6/2 + 1 = 0 \checkmark$$
$$\int_0^1 v_2v_3\,dx = \int_0^1 (12x^3-18x^2+8x-1)\,dx = \tfrac{12}{4}-\tfrac{18}{3}+\tfrac{8}{2}-1 = 0 \checkmark$$

$$\begin{array}{c|ccc} & 6 & -6 & 1 \\ \hline 2 & 12 & -12 & 2 \\ -1 & -6 & 6 & -1 \end{array}$$

[3] By least squares, find the equation of the form $z = ax + by + c$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We wish we could solve $Ax=b$:

$$\begin{array}{c} x \quad y \quad 1 \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{array}{c} z \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

Solve instead $A^TAx = A^Tb$:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$$

2	1	2	2	1
1	2	2	1	2
2	2	4	2	2
2	1	2	2	1
1	2	2	1	2

inverse

$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 3 \end{bmatrix} \Big/ 4 \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 3 \end{bmatrix} \Big/ 4 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix} \Big/ 4$$

So $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix} \Big/ 4$ best fit is $\boxed{-\frac{1}{2}x - \frac{1}{2}y + 5/4}$ answer

Check:

x	y	1	z	$-\frac12 x - \frac12 y + 5/4$	error
0	0	1	1	5/4	1/4
1	0	1	1	3/4	-1/4
0	1	1	1	3/4	-1/4
1	1	1	0	1/4	1/4

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \Big/ 4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

error is $\perp$ to $x, y, 1$ ✓

[4] Let V be the subspace of $\mathbb{R}^4$ defined by the system of equations

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} -1 & 2 & -1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{1} & \cancel{-1} \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2 = 1 + 3$
so rank 2
V is 2-dimensional

Find the 4×4 matrix A that projects $\mathbb{R}^4$ orthogonally onto V.

1 Find a basis for V: $\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 If every point was actually in V, we'd be able to solve $Ax=b$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

Instead solve $A^TAx = A^Tb$ to find nearest point in V:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1} = \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \Big/ 20 = \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \Big/ 10 \quad \checkmark$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \Big/ 10 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & 4 & 1 & -2 \\ -3 & -1 & 1 & 3 \end{bmatrix} \Big/ 10 \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

projection $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 7 & 4 & 1 & -2 \\ -3 & -1 & 1 & 3 \end{bmatrix} \Big/ 10 \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \boxed{\begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix} \Big/ 10} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$

answer

check:

$$\begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix} \Big/ 10 \; \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}}^{\text{in V}} & \overbrace{\begin{matrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{matrix}}^{\perp \text{ to V}} \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}}^{\text{same}} & \overbrace{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}}^{\text{zero}} \end{bmatrix}$$

$\checkmark \; \checkmark \; \checkmark \; \checkmark$

[...4] V is the set of solutions to our system of equations. Many people projected instead to the orthogonal complement $V^\perp$ of V, spanned by the rows of our matrix.

Call this projection B; $I = A + B$ so we can find B:

$$B = I - A = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \Big/ 10 - \begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix} \Big/ 10 = \boxed{\begin{bmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{bmatrix} \Big/ 10}$$

right answer to wrong question

We can also compute as before:

We can't solve $\begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ for all $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$

so we instead solve

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

This is Theorem 5.4.7 on p241 of Bretscher, that many people found and used. Unfortunately, it is easy to misapply a theorem one doesn't understand. It is the same work starting from scratch, with $A^TAx = A^Tb$ as our only tool.

$$\begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \Big/ 10 \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 4 & 1 & -2 \\ -2 & 1 & 4 & -3 \end{bmatrix} \Big/ 10 = \boxed{\begin{bmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{bmatrix} \Big/ 10}$$

right answer to wrong question

Note that Theorem 5.4.7 requires that the columns form a basis. Our original matrix was not full rank, so the formula fails if one doesn't fix this first by deleting a row.

[5] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\underset{1}{[\;]} \qquad \underset{1}{\begin{bmatrix} 1 \end{bmatrix}} \qquad \underset{1}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \qquad \underset{2}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}} \qquad \underset{3}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \qquad \underset{4}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}}$$

Find $f(3)$, $f(4)$, and $f(5)$. Find a recurrence relation for $f(n)$.

Expand by minors down first column:

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = \boxed{2 = f(3)} \;*$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 1\underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}}_{f(3)} + 1\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & \boxed{1} \end{vmatrix}}_{f(1)?} = 2+1 = \boxed{3 = f(4)} \;*$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{vmatrix} = 1\underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix}}_{f(4)} + 1\underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix}}_{f(2)?} = 3+1 = \boxed{4 = f(5)} \;*$$

general case:

$$\underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ & & & \vdots & & \vdots & \end{vmatrix}}_{f(n)} = \underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ & & & \vdots & & \vdots & \end{vmatrix}}_{f(n-1)} + \underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ & & & \vdots & & \vdots & \end{vmatrix}}_{f(n-3)}$$

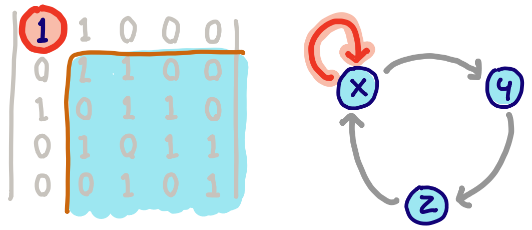
$$\boxed{f(n) = f(n-1) + f(n-3)} \;* \text{ answers}$$

check:

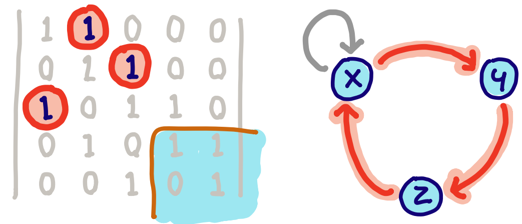
n	0	1	2	3	4	5	6	7	8
$f(n)$	1	1	1	2	3	4	6	9	$\boxed{13}$

check on next page

This is a similar pattern to the Fibonacci sequence.
Here, we choose patterns for the determinant one or three spots at a time.

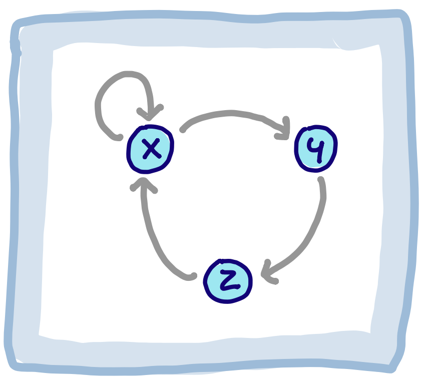


Choose one spot



Choose three spots

This reminds us of path counting on the graph below, where we choose paths ⓧ to ⓧ one or three steps at a time:



We can use this to check F(8) above.

$$
\begin{array}{cc} & \text{from} \\ & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \text{to} & \end{array}
$$

$$
\underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} = \underset{A^2}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}
$$

$$
\underset{A^2}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \underset{A^2}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \underset{A^4}{\begin{bmatrix} 3 & 1 & 2 \\ 2 & & \\ 1 & & \end{bmatrix}}
$$

$$
\underset{A^4}{\begin{bmatrix} 3 & 1 & 2 \\ & & \\ & & \end{bmatrix}} \underset{A^4}{\begin{bmatrix} 3 & & \\ 2 & & \\ 1 & & \end{bmatrix}} = \underset{A^8}{\begin{bmatrix} 13 & & \\ & & \\ & & \end{bmatrix}}
$$

☑