db89

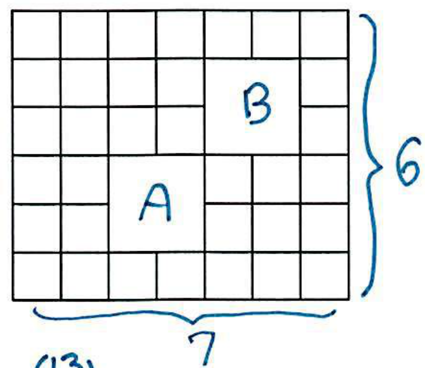
○ 0 ○ 1 ○ 2 ○ 3 ○ 4 ○ 5 ◉ 6

S14 Final Exam Problem 1
Combinatorics, Dave Bayer

Dave Bayer (Instructor)

db89e4p1

[1] How many paths are there from the lower left corner to the upper right corner of this grid, moving only up or to the right?



A at $(3,2)$
B at $(5,4)$

all paths $= \binom{7+6}{6} = \binom{13}{6}$

through A $= \binom{3+2}{3}\binom{4+4}{4} = \binom{5}{2}\binom{8}{4}$

through B $= \binom{5+4}{4}\binom{2+2}{2} = \binom{9}{4}\binom{4}{2}$

through A and B $= \binom{5}{2}\binom{4}{2}\binom{4}{2}$

Inclusion-exclusion = All − A − B + AB

$$\boxed{\binom{13}{6} - \binom{5}{2}\binom{8}{4} - \binom{9}{4}\binom{4}{2} + \binom{5}{2}\binom{4}{2}\binom{4}{2}}$$

$$\binom{13}{6} = \frac{13\cdot\cancel{12}^{2}\cdot 11\cdot\cancel{10}^{3}\cdot\cancel{9}\cdot\cancel{8}^{2}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 13\cdot 11\cdot 2\cdot 3\cdot 2 = 143\cdot 12$$

$$\begin{array}{r} 1430 \\ 286 \\ \hline 1716 \end{array}$$

$$\binom{5}{2}\binom{8}{4} = 10\cdot\frac{\cancel{8}\cdot 7\cdot\cancel{6}^{2}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 10\cdot 70 = 700$$

$$\binom{9}{4}\binom{4}{2} = \frac{9\cdot\cancel{8}\cdot 7\cdot\cancel{6}^{2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}\cdot 6 = 9\cdot 14\cdot 6 = 126\cdot 6 = 378\cdot 2 = 756$$

$$\binom{5}{2}\binom{4}{2}\binom{4}{2} = 10\cdot 6\cdot 6 = 360$$

$$\begin{array}{r} 1716 \\ -1456 \\ \hline 260 \end{array} + 360 = \boxed{620}$$

[1] continued.

1	7	28	74	160	246	376	620
1	6	21	46	86	86	130	244
1	5	15	25	40	B	44	114
1	4	10	10	15	26	44	70
1	3	6	A	5	11	18	26
1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1

$\boxed{\text{620 paths}}$ ✓

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0 1 2 3 4 5 6

S14 Final Exam Problem 2
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[2] How many ways can one fill a tube of length 16, using sticks of length 3 and 4?

$$f(n) = \underbrace{f(n-3)}_{\text{start with 3}} + \underbrace{f(n-4)}_{\text{start with 4}}$$

0	1
1	0
2	0
3	1
4	1
5	0
6	1
7	2
8	1
9	1
10	3
11	3
12	2
13	4
14	6
15	5
16	6

add pairs in sliding window

$\boxed{6 \text{ ways}}$

check:

12 = 3333
444

13 = 3334
3343
3433
4333

16 = 3 3334
3 3343
3 3433
3 4333
4 3333
4 444

S14 Final Exam Problem 3
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db89e4p3

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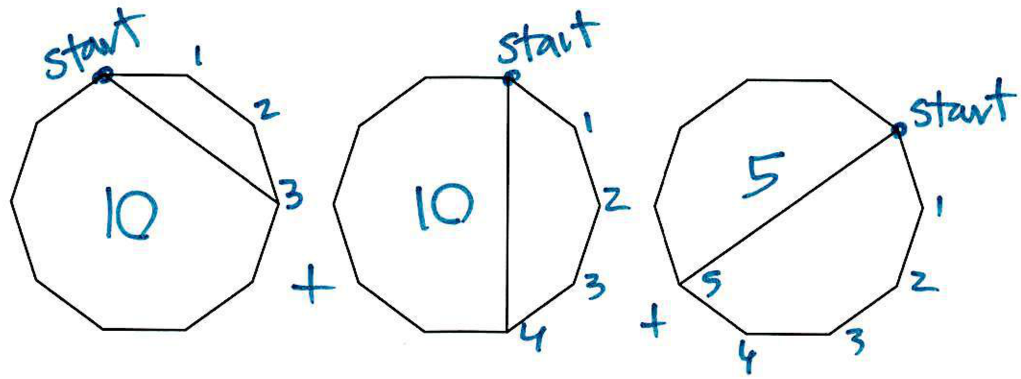
0 1 2 3 4 5 6

[3] How many ways can a 10-gon be dissected into two pieces, neither of which is a triangle?



$\binom{10}{2} = \frac{10\cdot 9}{2\cdot 1} = 5\cdot 9 = 45$ pairs of vertices

-10 adjacent (sides of n-gon)

-10 skip 1 (forms triangles)

$\boxed{25}$

check: 10 skip 3
10 skip 4
5 skip 5 (divide by 2, can't tell ends apart)

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0 1 2 3 4 5 6

S14 Final Exam Problem 4
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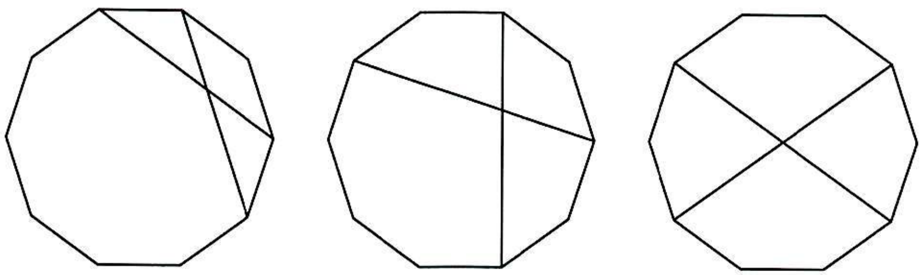
db89e4p4

[4] How many ways can one draw two crossing interior edges, inside a 10-gon?



$\binom{10}{4}$ 4-element subsets $\quad \dfrac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = \boxed{210}$

3 ways to pair each subset
exactly one gives a cross

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S14 Final Exam Problem 5
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db89e4p5

0 1 2 3 4 5 6

[5] How many standard Young tableaux are there of the following shape?

6	5	2	1
3	2		
2	1		

$8!/\prod(\text{hook lengths})$

$$\frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{6}\cdot \cancel{5}\cdot \cancel{2}\cdot 1\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{2}\cdot \cancel{1}} = \boxed{56}$$

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S14 Final Exam Problem 6
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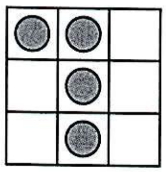
db89e4p6

0 1 2 3 4 5 6

[6] How many ways can 4 checkers be placed on a 3 × 3 checkerboard, up to symmetry? Consider both rotations and flips.



a	b	a
b		b
a	b	a

a	b	c
d		d
c	b	a

a	d	a
b	e	b
c	f	c

d	a	b
a	e	c
b	c	f

$\frac{1}{|G|}\sum_g |X^g|$ = average number of fixed patterns

in all, $\binom{9}{4} = \frac{9\cdot\cancel{8}\cdot 7\cdot \cancel{6}^{\,2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 126$ raw patterns

g	$\lvert X^g\rvert$	
1	126	
$\curvearrowright$	2	a or b
$\curvearrowleft$	2	
$\circlearrowright$	6	$\binom{4}{2}$ choose 2 of a,b,c,d
$\leftrightarrow$	12	$\binom{3}{2} + \binom{3}{1}\binom{3}{2} = 3 + 3\cdot 3 = 12$
$\updownarrow$	12	choose 2 of a,b,c
$\nearrow\!\swarrow$	12	or choose 1 of a,b,c
$\nwarrow\!\searrow$	12	and 2 of d,e,f

$136 + 48 = 184 \div 8 = \frac{160}{8} + \frac{24}{8}$

$= 20 + 3$

$\boxed{23}$

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○	○	○	○	○	○	◉
0	1	2	3	4	5	6

S14 Final Exam Problem 7
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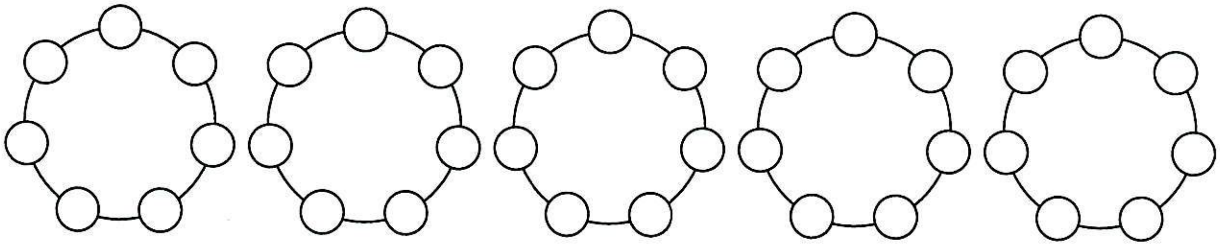
db89e4p7

[7] Up to symmetry, how many ways can the beads of a seven bead necklace be colored, using exactly three colors? Consider only rotations, and use all three colors.



$\frac{1}{|G|}\sum_g |X^g|$

using at most n colors:

n	$(n^7+6n)/7$
1	1
2	20
3	315

$(128+12)/7 = 20$

$3^4 = 9^2 = 81$

$3^7 = 27 \cdot 81$

$$\begin{array}{r} 27 \\ 81 \\ \hline 27 \\ 216 \\ \hline 2187 \end{array}$$

$$\begin{array}{r} 2187 \\ 18 \\ \hline 2205 \end{array} \div 7$$

315

RGB−RG−RB−GB
+R+G+B

Inclusion/exclusion

$315 - 3 \cdot 20 + 3 = \boxed{258}$

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S14 Final Exam Problem 8
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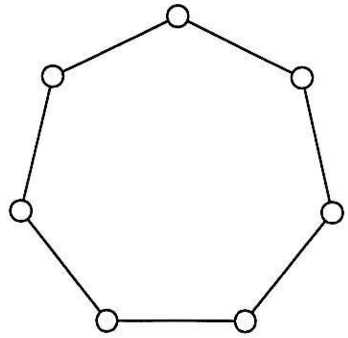
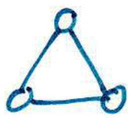
db89e4p8

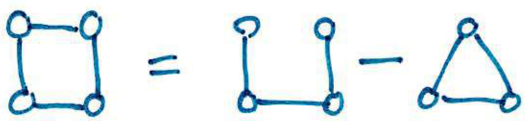
0 1 2 3 4 5 6

[8] Using n possible colors, how many ways can one properly color the vertices of a 7-gon?



Warm up: 

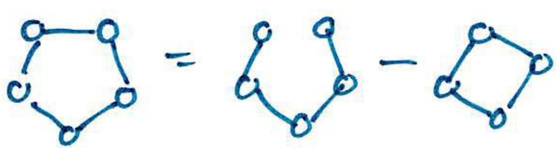
$$n(n-1)(n-2) = n^3 - 3n^2 + 2n = n^3 - 3n^2 + 3n - 1 - (n-1) = (n-1)^3 - (n-1)$$



$$= n(n-1)^3 - (n-1)^3 + (n-1)$$

deletion/contraction

$$= (n-1)^4 + (n-1)$$



$$= n(n-1)^4 - (n-1)^4 - (n-1) = (n-1)^5 - (n-1)$$

k-gon	
3	$(n-1)^3 - (n-1)$
4	$(n-1)^4 + (n-1)$
5	$(n-1)^5 - (n-1)$
6	$(n-1)^6 + (n-1)$
7	$\boxed{(n-1)^7 - (n-1)}$

by induction,...

[8] continued...

Matroid approach

inclusion/exclusion on subsets of edges

k vertices, subset S of edges

$$\sum (-1)^{|S|} \underbrace{n^{k-\mathrm{rank}(S)}}_{\text{colorings which } \underline{\text{agree}} \text{ on ends of each edge}}$$

For k-gon, only entire edge set $S=E$ has rank $<$ size

so

$$\sum (-1)^{|S|} n^{k-\mathrm{rank}(S)} = (n-1)^k + \text{correction for } S=E$$

$(-1)^k$ should be $(-1)^k n$

$$\boxed{(n-1)^k + (-1)^k (n-1)}$$

for $k=7$, $\boxed{(n-1)^7 - (n-1)}$