Homework 1 Solutions

①

A) set up your A' matrix: "1" if possible path
"0" if not possible

		out	
	x	y	z
in x			
y			
z			

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A'$$

$$A^2 = (A')^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^3 = (A^2)(A') = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A^5 = (A^3)(A^2) = \begin{bmatrix} 1 & 6 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 5 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$A^{10} = (A^5)(A^5) = \begin{bmatrix} 1 & 15 & 5 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 15 & 5 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}.$$

"in x, out y" translates to "you want the value of the x^{th} row and y^{th} column".

this value!

$$A^{10} = \begin{bmatrix} & * & \\ & & \\ & & \end{bmatrix}$$

$$* = (1\ 15\ 5)\cdot(15\ 1\ 5) = 15+15+25$$
$$= \boxed{55}$$

2

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Approach to row-reducing: you want to isolate $w, x, y,$ and z by ultimately making the problem look like:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$$

where a, b, c, d are constants and your desired solutions.

First, we look @ the value in the 2^{nd} row and 1^{st} column and try to make it zero by adding the first row to 2(row2): ① + 2② → ②

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Now, we look at the value in 3^{rd} row, 2^{nd} column and try to make this zero: add row 2 to 3(row 3).

② + 3③ → ③

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Now we look at this value and try to make it zero:

row3 + 4(row4)

$③ + 4④ \longrightarrow ④$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$$

At this point you have an "upper triangular" matrix meaning that every value below the diagonal is zero. You may stop row reducing now or you can continue to get $\left[I \middle| \begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix} \right]$ form as originally planned.

If you stop row reducing now:

- $5z = 20$ so $z = 4$

(from last row)

- $4y - 3z = 0$ (from row3)

$4y - 3(4) = 0$ so $y = 3$

If you continue row reduction method:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$$

we want these 3 values to be made zero in the same approach we took before!

First we try to make this value zero.

(stopped row reducing)

- $3x - 2y = 0$ (from row2)

  $3x - 2(3) = 0$ so $x = 2$

- $2w - x = 0$ (from row1)

  $2w - 2 = 0$ so $w = 1$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(Continued row reducing)

5(row3) + 3(row4):

$5(3) + 3(4) \rightarrow (3)$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 20 & 0 & 60 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$$

Now we make this zero

10(row2) + row3:

$10(2) + (3) \rightarrow (2)$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 & 60 \\ 0 & 0 & 20 & 0 & 60 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$$

Now we make this zero:

30(row1) + row2:

$30(1) + (2) \rightarrow (1)$

$$\left[\begin{array}{cccc|c} 60 & 0 & 0 & 0 & 60 \\ 0 & 30 & 0 & 0 & 60 \\ 0 & 0 & 20 & 0 & 60 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$$

Now divide row1 through by 60
divide row2 through by 30
divide row3 through by 20

divide row 4 through by 5.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

so $w = 1$
$x = 2$
$y = 3$
$z = 4$.

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③ We attack using row reduction, just like problem 2:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 3 & 2 \\ 3 & 0 & 1 & 4 & 4 \end{array}\right]$$

$\downarrow$ ② − 2① → ②

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 1 & -2 \\ 3 & 0 & 1 & 4 & 4 \end{array}\right]$$

$\downarrow$ ③ − 3① → ③

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 0^{*} & -2^{\star} & 1 & -2 \\ 0 & 0 & -2 & 1 & -2 \end{array}\right].$$

You can't get a pivot at the * position b/c it's already 0 and you can't get one from zero. So, the pivot is at ★.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -2^{\star} & 1 & -2 \\ 0 & 0 & -2 & 1 & -2 \end{array}\right]$$

Because $\star$ is your pivot you now want the value below this to be made zero: ② − ③ → ③

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ this is meaningless so you can just cross it out!

What the row of 0's tells you is that in your original problem, you had a row that was merely a linear combination of the other two rows!

Notice that in the original problem, ③ = ① + ②!

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 1 & -2 \end{array}\right]$$

$-\frac{1}{2}$ ② → ②

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$$

Now you have a 1 at each pivot position as desired, but what about the two columns w/out pivots? Those are essentially "free" columns.

① − ② → ①

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{3}{2} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 \end{array}\right] \\ \begin{array}{cccc} w & x & y & z \\ & \text{free} & & \text{free} \end{array} \end{array}$$

So in your particular solution, you can just set them (x and z) equal to zero arbitrarily:

$x = 0$

$z = 0$

This makes

$w = 1$

$y = 1$

So one particular solution is $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Now, the homogeneous solution should have <u>two</u> dimensions because you started out with 4 variables (w, x, y, z) corresponding to the # of columns, but you only had 2 independent equations (corresponding to the # of rows) and $4-2=2$.

So in general: # variables − # independent equations = # dimensions in your homogeneous solution.

<u>Or</u> another way to say this is:

columns − # rows = # dim in homog. sol'n.

given that the rows are independent!

Ok, so how do you get the actual homogeneous solution? Since x and z are both free, we let one remain zero and look at the effect of changing the other on the rest of the variables.

Let's start by keeping x as zero: (and changing z)

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/2 & 1 \\ 0 & 0 & 1 & -1/2 & 1 \end{array}\right] \\ \begin{array}{cccc} w & x & y & z \end{array} \end{array} \longrightarrow$$

we look at the row 2: (easier than row 1)

$$y - \tfrac{1}{2}z = 1.$$

when $z=0$, $y=1$

when $z=1$, $y=3/2$

when $z=2$, $y=2$
when $z=3$, $y=5/2$

As z increases by 1,
y increases by 1/2.

Now we look at row 1:

$$w + \tfrac{3}{2}z = 1$$

when $z=0$, $w=1$
$z=1$, $w=-1/2$
$z=2$, $w=-2$
$z=3$, $w=-7/2$

As z increases by 1
w decreases by 3/2.

So part of your homogeneous solutions looks like this:

$$\begin{bmatrix} -3/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} s$$

*remember we kept $x=0$ to examine z alone in its effect on the other variables.

Now, we keep $z=0$ and examine the effect of changing x on the other variables:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \cancel{3/2} & 1 \\ 0 & 0 & 1 & \cancel{-1/2} & 1 \end{array}\right]$$

when $z=0$, these two essentially go to zero when multiplied by z.

From row 1: $w + 0x = 1 \longrightarrow w=1$
$y + 0x = 1 \longrightarrow y=1$

Changing x does not change w or y.
In other words, as x increases by {1, 2, 3, etc.} $\rightarrow$ w changes by 0. $\rightarrow$ y changes by 0.

So part of your homogeneous solutions looks like,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} t$$

* remember we kept $z=0$ this time.

Now let's combine the homogeneous solutions and the particular to get a final answer:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} t$$

Since "s" is just a parameter multiplying the vector associated with it by a scalar won't change anything. So, to get rid of fractions:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} t$$

or

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$