Homework 2 Solutions

1) Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 1 & 7 \\ 1 & 1 \end{bmatrix}$$

subtract 7 (second row) from the first row

multiply first row by $-\frac{1}{6}$

$$\begin{bmatrix} 1 & 7 \\ 1 & 1 \end{bmatrix} \xrightarrow{① \leftarrow ① - 7②} \begin{bmatrix} -6 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{① \leftarrow -\frac{1}{6}①} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{② \leftarrow ② - ①} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Perform row operations to get to identity matrix

Then, do the opposite and rewrite as matrices

$① \leftarrow ① - 7②$ $\begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$ $\quad ① \leftarrow -\frac{1}{6}①$ $\begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix}$ $\quad ② \leftarrow ② - ①$ $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\Big\}$ what you did to get from A to I

opposite steps

inverses of each other

$① \leftarrow ① + 7②$ $\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$ $\quad ① \leftarrow -6①$ $\begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad ② \leftarrow ② + 1$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\Big\}$ the opposite of above (matrices that represent are the inverse)

can check: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Big/ ad-bc$

$$\begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{6} \end{bmatrix} \Big/ -\tfrac{1}{6} = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$$

To check, multiply from right to left

$$\begin{bmatrix} -6 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 7 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 1 & 1 \end{bmatrix} \checkmark$$

* there's more than one possible solution

☆ elementary matrix is one step away from identity, i.e. differs by one elementary row operation

e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ yes!

$\begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$ yes!

$\begin{bmatrix} 1 & 0 \\ 1000 & 1 \end{bmatrix}$ yes!

$\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ no ☹

2) Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w\\ x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\\ 1\end{bmatrix} + \begin{bmatrix}0&3\\ 1&2\\ 2&1\\ 3&0\end{bmatrix}\begin{bmatrix}s\\ t\end{bmatrix} \leftarrow$$

i.e. $\begin{bmatrix}1\\ 1\\ 0\\ 1\end{bmatrix}$ is the particular solution and $\begin{bmatrix}0&3\\ 1&2\\ 2&1\\ 3&0\end{bmatrix}$ is the homogenous solution

First, find a matrix that mult. with $\begin{bmatrix}0&3\\ 1&2\\ 2&1\\ 3&0\end{bmatrix}$ to give 0

$$\begin{bmatrix}0&1&2&3\\ 3&2&1&0\end{bmatrix}\begin{bmatrix} \quad ? \quad \end{bmatrix} = 0$$

One way to find this is to look at $\begin{bmatrix}0&\boxed{1\ 2}&3\\ 3&\boxed{2\ 1}&0\end{bmatrix}$ and decide that this part should be multiplied by the identity.

$$\begin{bmatrix}0&1&2&3\\ 3&2&1&0\end{bmatrix}\begin{bmatrix} & \\ 1&0\\ 0&1\\ & \end{bmatrix} = 0$$

Now, to find the other values, what numbers work that would satisfy both rows and make them multiply to get 0?

e.g. $3\cdot(?) + 2\cdot 1 + 1\cdot 0 + 0\cdot(?) = 0$

$? = -\frac{2}{3}$

$$\begin{bmatrix}0&1&2&3\\ 3&2&1&0\end{bmatrix}\begin{bmatrix} -\frac{2}{3} & \\ 1&0\\ 0&1\\ & -\frac{1}{3} \end{bmatrix} = 0$$

and similarly,

$$\begin{bmatrix}0&1&2&3\\ 3&2&1&0\end{bmatrix}\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3}\\ 1&0\\ 0&1\\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} = 0$$

So one such complementary matrix would be

$$\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3}\\ 1&0\\ 0&1\\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Multiply by -3 so it looks nice (it's okay to do because $\begin{bmatrix}0&3\\ 1&2\\ 2&1\\ 3&0\end{bmatrix}$ is homog. and mult. by s, t) $\rightarrow$ $\begin{bmatrix}2&1\\ -3&0\\ 0&-3\\ 1&2\end{bmatrix}$ $\leftarrow$ these are still complementary

$0\cdot -3 = 0$

$$\begin{bmatrix}2&-3&0&1\\ 1&0&-3&2\end{bmatrix}\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix} = ???$$

This means

$$\begin{bmatrix}2&-3&0&1\\ 1&0&-3&2\end{bmatrix}\left(\begin{bmatrix}1\\ 1\\ 0\\ 1\end{bmatrix} + \underbrace{\cancel{\begin{bmatrix}0&3\\ 1&2\\ 2&1\\ 3&0\end{bmatrix}}}_{\text{goes to } 0}\begin{bmatrix}s\\ t\end{bmatrix}\right) = \begin{bmatrix}2&-3&0&1\\ 1&0&-3&2\end{bmatrix}\begin{bmatrix}1\\ 1\\ 0\\ 1\end{bmatrix} = \text{right hand side}$$

(continued) $\rightarrow$

$$\begin{bmatrix} 2 & -3 & 0 & 1 \\ 1 & 0 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

↳ a solution:

$$\boxed{\begin{bmatrix} 2 & -3 & 0 & 1 \\ 1 & 0 & -3 & 2 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

Note: there are many more possible solutions e.g. for the matrix complementary to $\begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$

you could choose

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}} = 0$$

just picked 2 here and then filled in what else was needed to work (sorry not systematic ☺)

* if your compl. slns are a linear comb. of these, then they're ok!

e.g.

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \quad ✓$$

* don't pick

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

because these two columns are the same, so don't give any new info →

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}s + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}t = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}r$$

same thing with

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -\frac{2}{3} & 2 \\ 1 & -3 \\ 0 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \quad \text{etc}$$

↳ then use that other matrix to find right hand side

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$$

3) Find the intersection of the following two affine subspaces of $\mathbb{R}^4$

① $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

② $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$

One method

Solve for system of equations that has ① as solution set, then plug ② in

Find compl. to homog.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} ? \end{bmatrix} = 0 \qquad \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 1 & 2 \end{matrix}} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ \begin{matrix} -1 & -2 \\ -1 & 0 \end{matrix} \end{bmatrix}}_{} = 0$$

* for more detailed method/other ways to find compl. sln, see problem #2

Right hand side

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

plug ② in here

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

→ multiply both sides by inverse of $\begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix}$

continued

$$\begin{bmatrix}-2 & 1\\ -3 & 0\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix} \qquad \begin{bmatrix}-2 & 1\\ -3 & 0\end{bmatrix}^{-1}=\begin{bmatrix}0 & -1\\ 3 & -2\end{bmatrix}\Big/_{-2\cdot 0-1\cdot -3}=\begin{bmatrix}0 & -1\\ 3 & -2\end{bmatrix}\Big/_{3}$$

$$\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}0 & -1\\ 3 & -2\end{bmatrix}\Big/_{3}\begin{bmatrix}1\\ 3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}-3\\ -3\end{bmatrix}=\begin{bmatrix}-1\\ -1\end{bmatrix}$$

$c=-1,\ d=-1$ → Plug these back into ②

$$\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 1\\ 1\\ 3\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\\ 3 & 0\end{bmatrix}\begin{bmatrix}-1\\ -1\end{bmatrix}$$

$$=\begin{bmatrix}2\\ 1\\ 1\\ 3\end{bmatrix}+\begin{bmatrix}-1\\ 0\\ -1\\ -3\end{bmatrix}+\begin{bmatrix}0\\ -1\\ 0\\ 0\end{bmatrix}=\boxed{\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix}}$$

Another method: solve for system of equations that has ② as solution set, then plug ① in

Find compl. matrix to $\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\\ 3 & 0\end{bmatrix}$

same steps as previous method

you'll end up with $a=-1,\ b=-1$, then plug into ① to get solution

$$\boxed{\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}}$$

✓

Yet another method

Set the two equal to each other

$$\begin{bmatrix}3\\2\\1\\1\end{bmatrix}+\begin{bmatrix}1&1\\2&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2\\1\\1\\3\end{bmatrix}+\begin{bmatrix}1&0\\0&1\\1&0\\3&0\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}$$

$$\begin{bmatrix}1&1&-1&0\\2&0&0&-1\\1&0&-1&0\\0&1&-3&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\\2\end{bmatrix}\Rightarrow\left[\begin{array}{cccc|c}1&1&-1&0&-1\\2&0&0&-1&-1\\1&0&-1&0&0\\0&1&-3&0&2\end{array}\right]$$

Then row reduce

$$② \leftarrow ② - 2① \quad \left[\begin{array}{cccc|c}1&1&-1&0&-1\\0&-2&2&-1&1\\1&0&-1&0&0\\0&1&-3&0&2\end{array}\right] \quad ③ \leftarrow ③ - ① \quad \left[\begin{array}{cccc|c}1&1&-1&0&-1\\0&-2&2&-1&1\\0&-1&0&0&1\\0&1&-3&0&2\end{array}\right] \quad ④ \leftarrow ④ + ③ \quad \left[\begin{array}{cccc|c}1&1&-1&0&-1\\0&-2&2&-1&1\\0&-1&0&0&1\\0&0&-3&0&3\end{array}\right]$$

$$② \leftarrow ② - 2③ \quad \left[\begin{array}{cccc|c}1&1&-1&0&-1\\0&0&2&-1&-1\\0&-1&0&0&1\\0&0&-3&0&3\end{array}\right] \xrightarrow{① \leftarrow ① + ③} \left[\begin{array}{cccc|c}1&0&-1&0&0\\0&0&2&-1&-1\\0&-1&0&0&1\\0&0&-3&0&3\end{array}\right] \quad ② \leftarrow ② + \tfrac{2}{3}④ \quad \left[\begin{array}{cccc|c}1&0&-1&0&0\\0&0&0&-1&1\\0&-1&0&0&1\\0&0&-3&0&3\end{array}\right]$$

$$\downarrow ① \leftarrow ① - \tfrac{1}{3}④$$

$$\left[\begin{array}{cccc|c}1&0&0&0&-1\\0&0&0&-1&1\\0&-1&0&0&1\\0&0&1&0&-1\end{array}\right] \xleftarrow{④ \leftarrow -\frac{1}{3}③} \left[\begin{array}{cccc|c}1&0&0&0&-1\\0&0&0&-1&1\\0&-1&0&0&1\\0&0&-3&0&3\end{array}\right]$$

$a=-1$
$-d=1$
$-b=1$
$c=-1$

$a=-1$, $b=-1$, $c=-1$, $d=-1$ then plug these back into ① or ②

$$① \quad \begin{bmatrix}w\\x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\2\\1\\1\end{bmatrix}+\begin{bmatrix}1&1\\2&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix}=\boxed{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}$$

$$② \quad \begin{bmatrix}w\\x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\1\\3\end{bmatrix}+\begin{bmatrix}1&0\\0&1\\1&0\\3&0\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix}=\boxed{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}$$