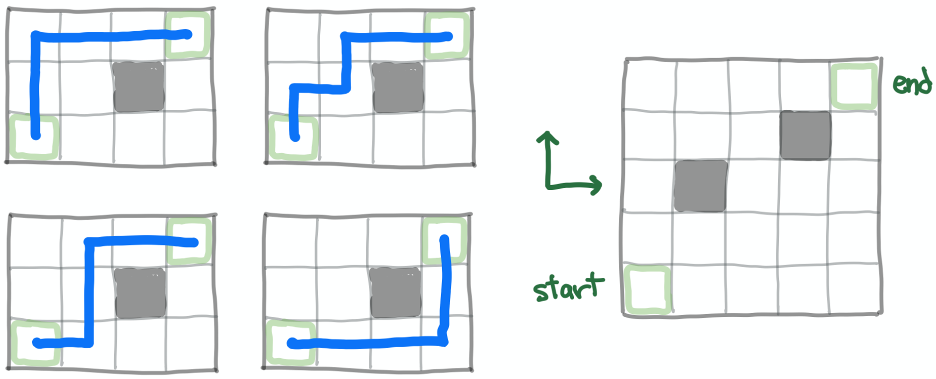
Exam 1

Combinatorics, Dave Bayer, February 11-14, 2021

[1] Moving up or over, for the grid on the left there are four paths between the corners that avoid the obstacle. For the grid on the right, how many paths avoid both obstacles?



1	2	6	6	18
1	1	4		12
1		3	7	12
1	2	3	4	5
1	1	1	1	1

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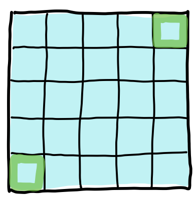

$$\binom{8}{4} = \frac{\cancel{8}\cdot 7\cdot \overset{2}{\cancel{6}}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 70$$

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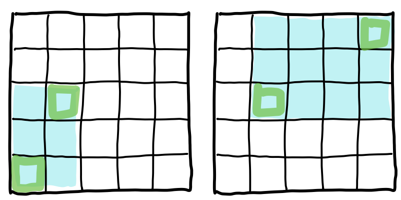

$$\binom{3}{1}\binom{5}{3} = 3\cdot 10 = 30$$

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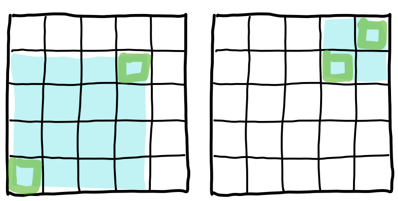
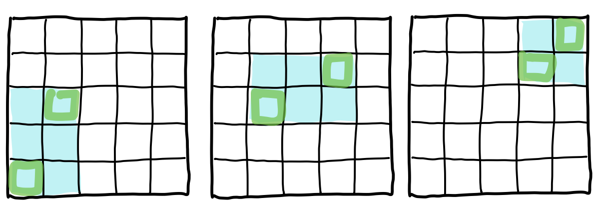


$$\binom{6}{3}\binom{2}{1} = 20\cdot 2 = 40$$

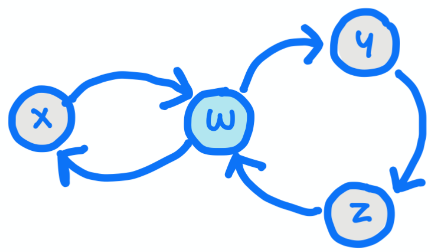
+

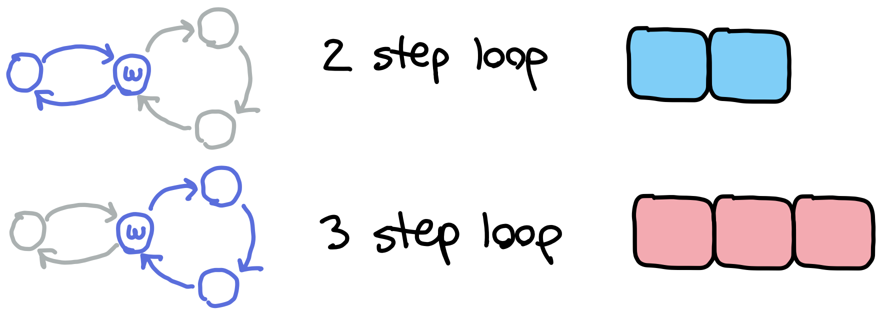


$$\binom{3}{1}\binom{3}{2}\binom{2}{1} = 18$$

$$70 - 30 - 40 + 18 = 18 \ \boxed{\checkmark}$$

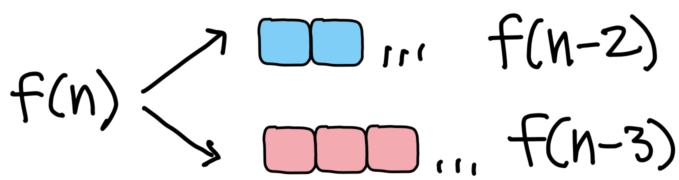
[2] Let $f(n)$ be the number of n step paths from w to itself on the directed graph below. What is $f(12)$?



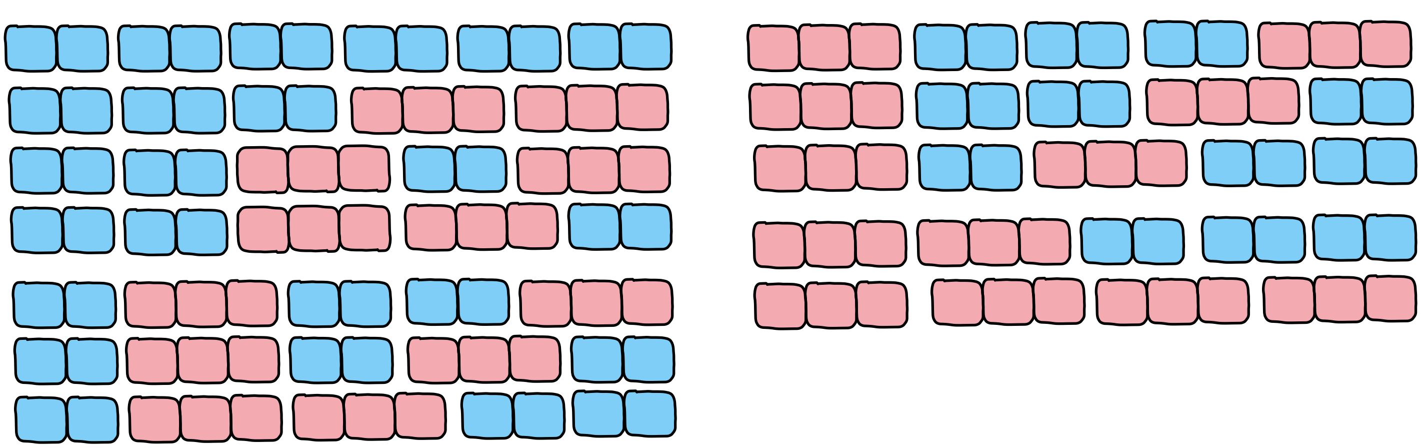


$\boxed{f(12) = 12}$

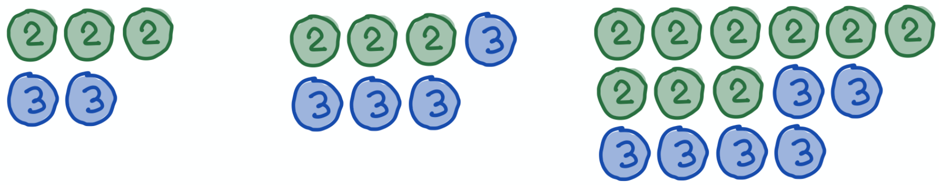
$$f(n) = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ f(n-2) + f(n-3), & n > 0 \end{cases}$$



n	0	1	2	3	4	5	6	7	8	9	10	11	12
f(n-2)			1		1	1	1	2	2	3	4	5	7
f(n-3)				1		1	1	1	2	2	3	4	5
f(n)	1		1	1	1	2	2	3	4	5	7	9	12

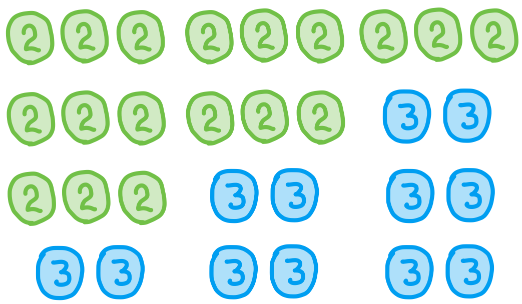


[3] Let $f(n)$ be the number of ways of making change for n cents, using 2 cent and 3 cent coins. As shown below, $f(6) = 2$, $f(9) = 2$, and $f(12) = 3$. What is $f(18)$?



Let $g(t) = \sum_{n=0}^{\infty} f(n)t^n$ be the generating function for $f(n)$. Find a closed form expression for $g(t)$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1		1		1		1		1		1		1		1		1		1
3	1		1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4



$$\boxed{f(18) = 4}$$

$$g(t) = 1 +$$

$$g(t) = (1 + \underbrace{t^2 + t^4 + t^6 + \ldots}_{1/1-t^2})(1 + \underbrace{t^3 + t^6 + t^9 + \ldots}_{1/1-t^3})$$

$$\boxed{g(t) = \frac{1}{1-t^2-t^3+t^5}}$$

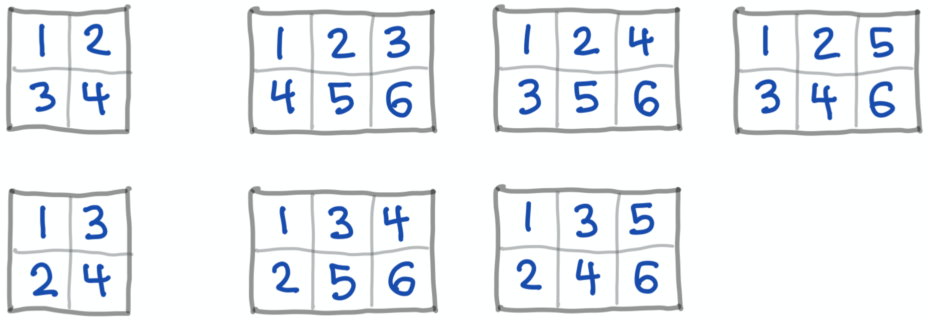
$$(1-t^2)(1-t^3) = 1 - t^2 - t^3 + t^5$$

check: $(1-t^2-t^3+t^5)\,g(t) = 1$

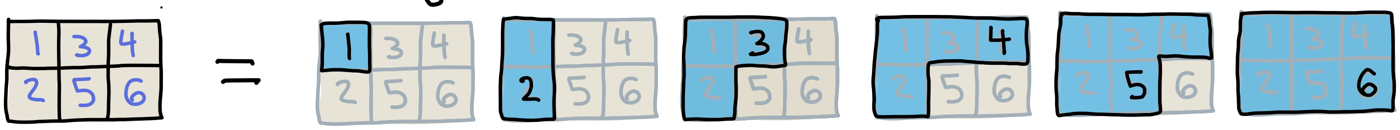
$g(t)$

1	1		1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4
$-t^2$			1		1	1	1	1	2	1	2	2	2	2	3	2	3	3	3
$-t^3$				1		1	1	1	1	2	1	2	2	2	2	3	2	3	3
$+t^5$						1		1	1	1	1	2	1	2	2	2	2	3	2
	1																		

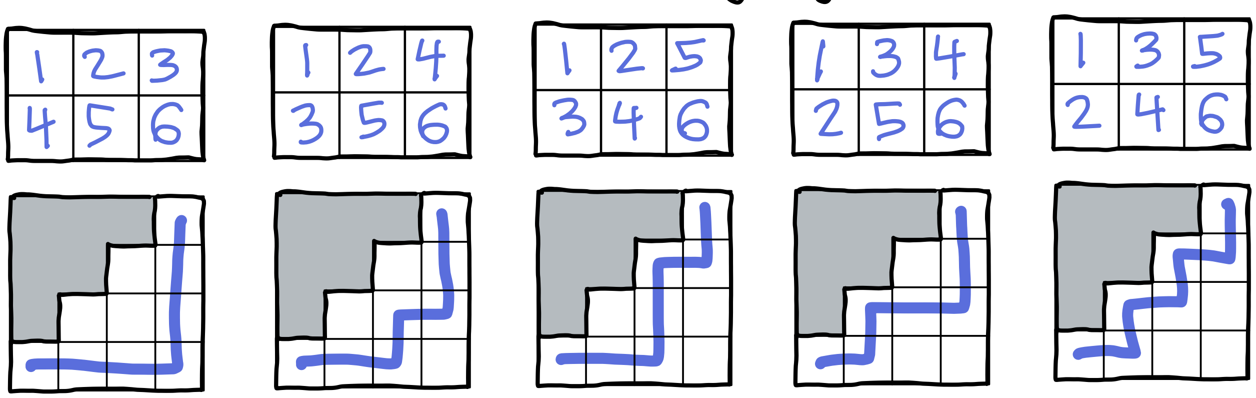
[4] A *Young tableau* is a way of filling in a staircase-shaped grid with the integers from 1 to n, so every row and every column is in ascending order. Let $f(n)$ be the number of Young tableaus for a $2 \times n$ grid. As shown below, $f(2) = 2$ and $f(3) = 5$. What is $f(5)$? What can you say about $f(n)$?



We can think of a Young tableau as instructions for growing a staircase:



The second row can't get ahead of the first row.
Think of first row as →, second row as ↑
This is same problem as not exceeding diagonal in a lattice walk.



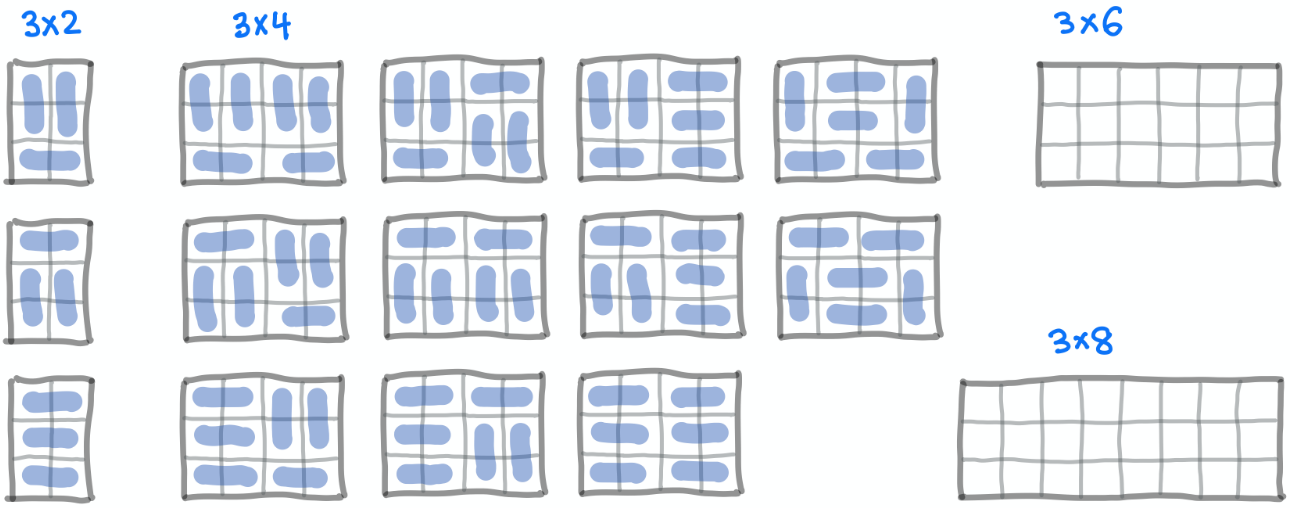
We recognize the Catalan numbers.

n	1	2	3	4	5	6
$f(n)$	1	2	5	14	42	132

$$\boxed{f(5) = 42}$$

$$\boxed{f(n) = C_n}$$

[5] Let $f(n)$ be the number of ways of arranging 1×2 bricks in a $3 \times 2n$ grid. As shown below, $f(1) = 3$ and $f(2) = 11$. Find $f(3)$ and $f(4)$. What can you say about $f(n)$?



$\boxed{f(3) = 41}$ $\boxed{f(4) = 153}$

See class notes, February 16