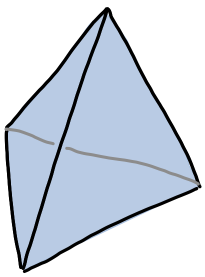
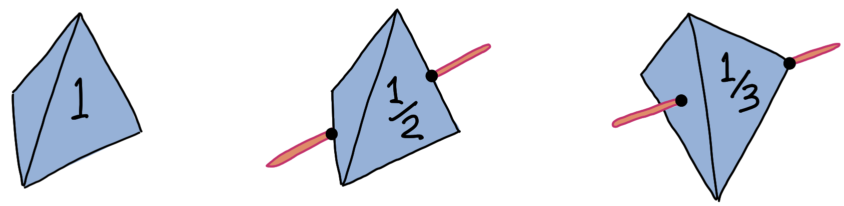
Exam 2

Combinatorics, Dave Bayer, March 18-21, 2021

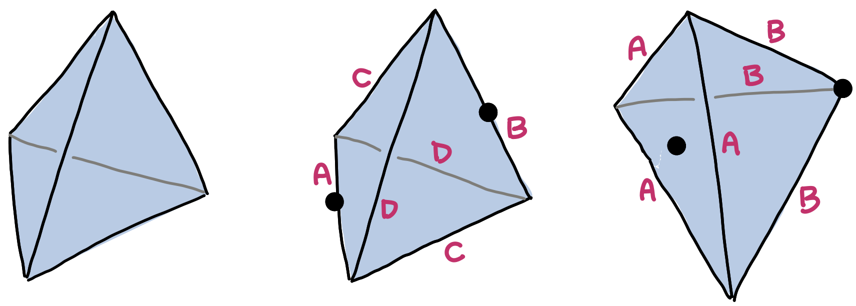
To receive full credit for correct answers, please show all work.

[1] How many ways can we choose three edges of a regular tetrahedron, up to rotational symmetry? Confirm your answer by finding all patterns up to symmetry.





1 3 8

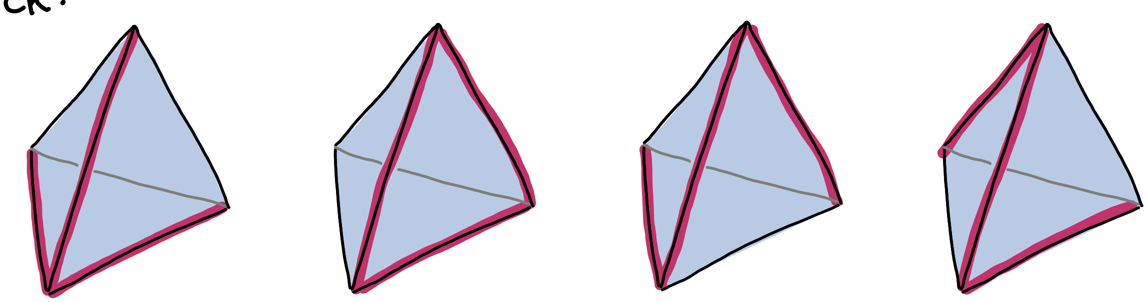


A B CC DD AAA BBB

$\binom{6}{3}=20$ $2\cdot 2=4$ 2

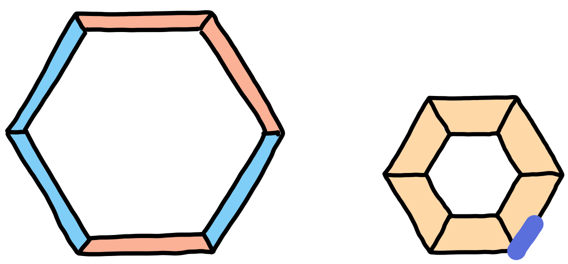
$$\frac{1}{12}\left(20+\underbrace{3\cdot 4}_{12}+\underbrace{8\cdot 2}_{16}\right) = 48/12 = \boxed{4}$$

Check:



Chiral pair

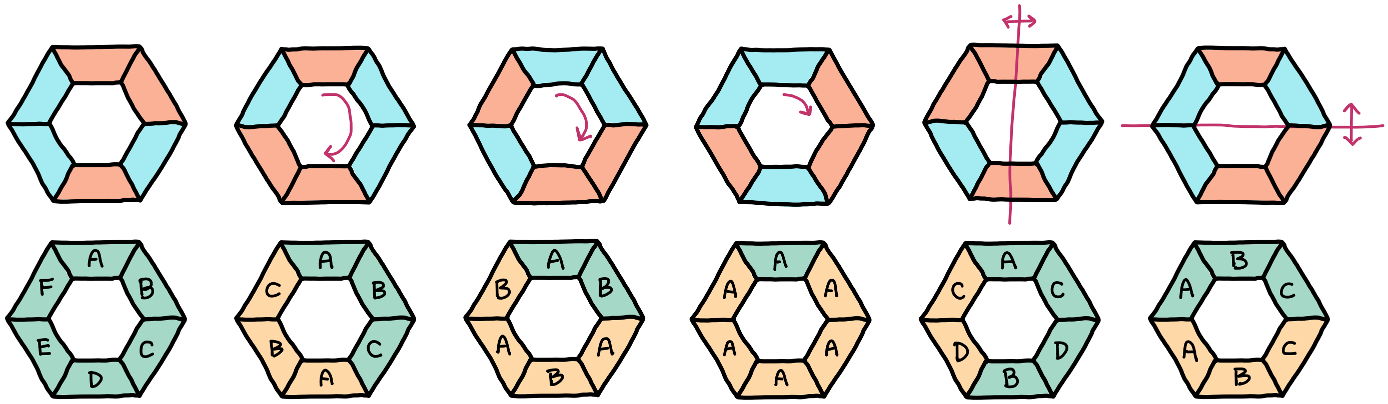
[2] How many ways can we k-color the six sides of a regular hexagon, up to rotational and flip symmetries? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.



$|G| = 6$ vertices $\cdot$ 2 edges $= 12$ cases / possibilities

6 rotations

6 flips

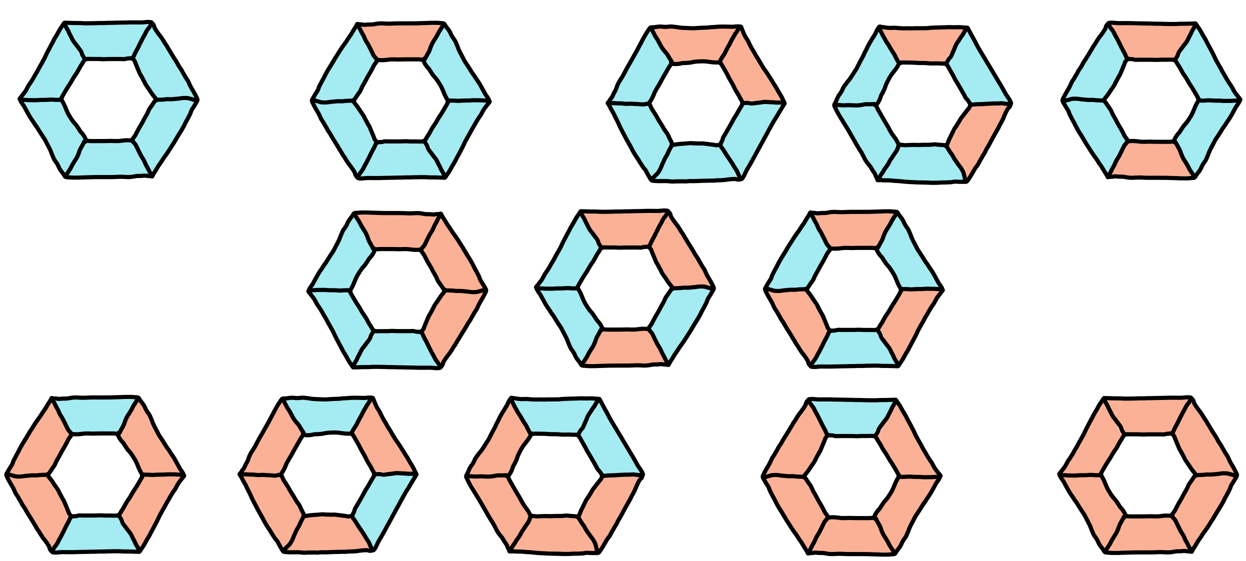


1	1	2	2	3	3
Identity	½ turn	⅓ turns	⅙ turns	side flips	vertex flips
k^6	k^3	k^2	k	k^4	k^3

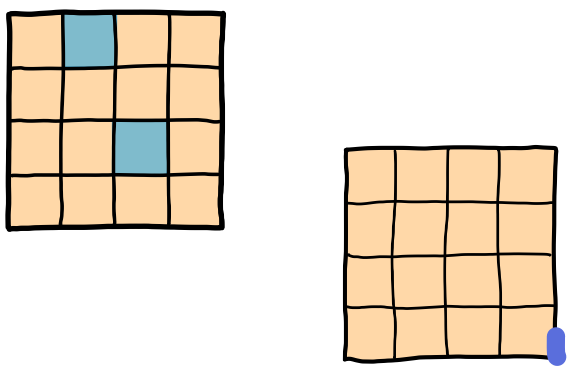
$$\boxed{\frac{1}{12}\left(k^6 + 3k^4 + 4k^3 + 2k^2 + 2k\right)}$$

$$k=2: \quad \frac{1}{12}\left(64 + \underset{48}{3\cdot 16} + \underset{32}{4\cdot 8} + \underset{8}{2\cdot 4} + \underset{4}{2\cdot 2}\right) = 156/12 = \boxed{13}$$

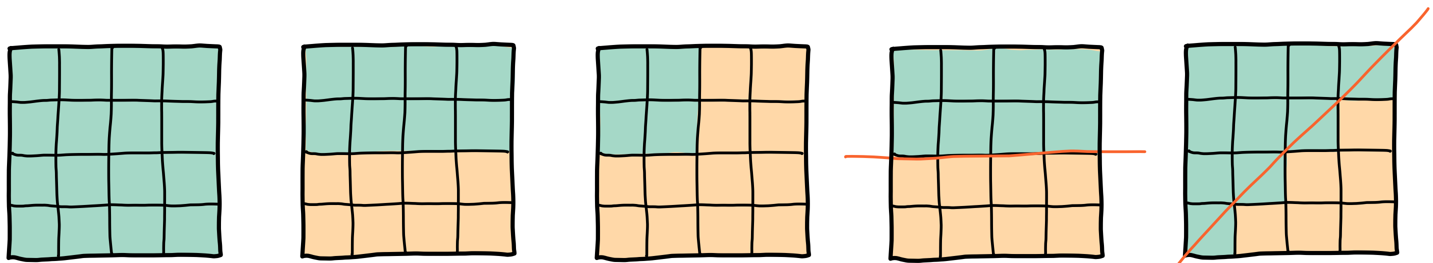
Check:



[3] How many ways can we choose two squares of a 4×4 board, up to rotational and flip symmetries? Confirm your answer by finding all patterns up to symmetry.



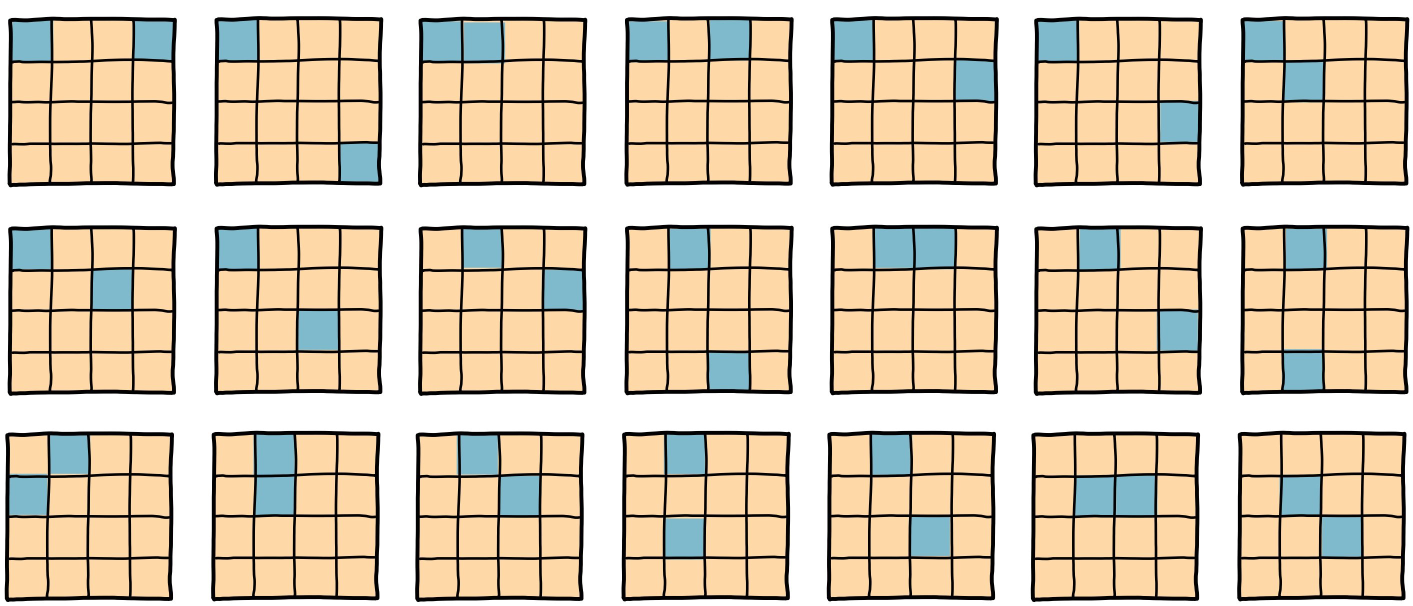
$$|G| = 4 \text{ corners} \cdot 2 \text{ edges} = 8$$



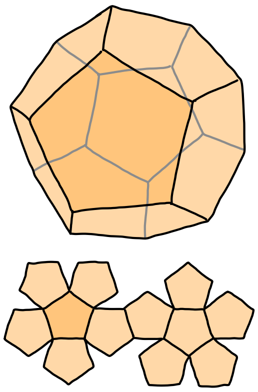
1	1	2	2	2
Identity	1/2 turn	1/4 turns	side flips	vertex flips
$\binom{16}{2} = 8 \cdot 15 = 120$	8	0	8	$\binom{4}{2} + 6 = 12$

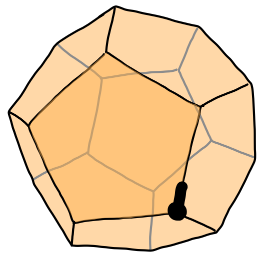
$$\frac{1}{8}\left(120 + 8 + \underbrace{2 \cdot 8}_{16} + \underbrace{2 \cdot 12}_{24}\right) = 168/8 = \boxed{21}$$

Check:



[4] How many ways can we choose 2 or 3 faces of a regular dodecahedron up to rotational symmetry? Confirm your answers by finding all patterns up to symmetry.





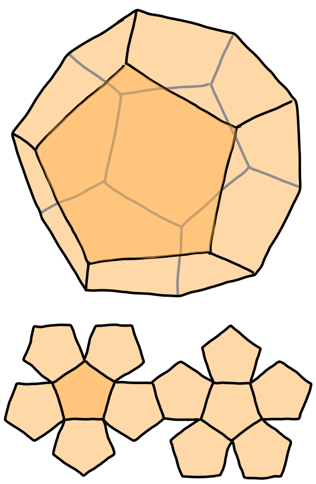
12 pentagon faces

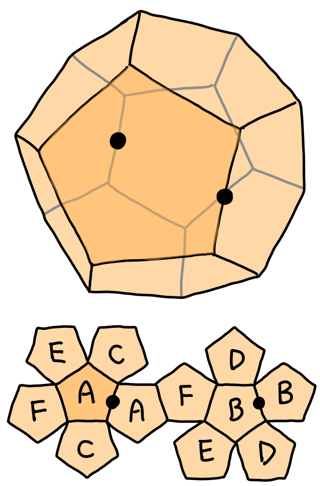
30 edges $5\cdot 12/2$

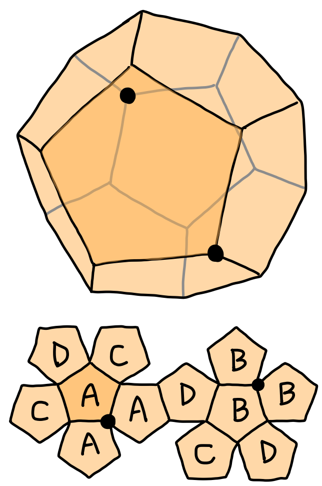
20 vertices $5\cdot 12/3$

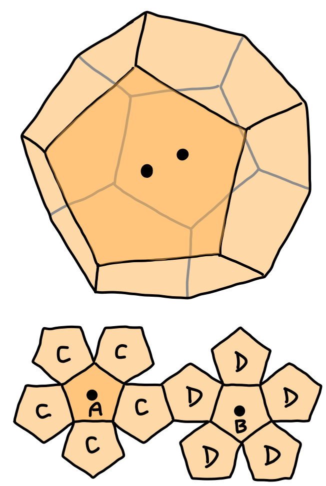
Choose vertex then edge

$|G| = 20\cdot 3 = 60$







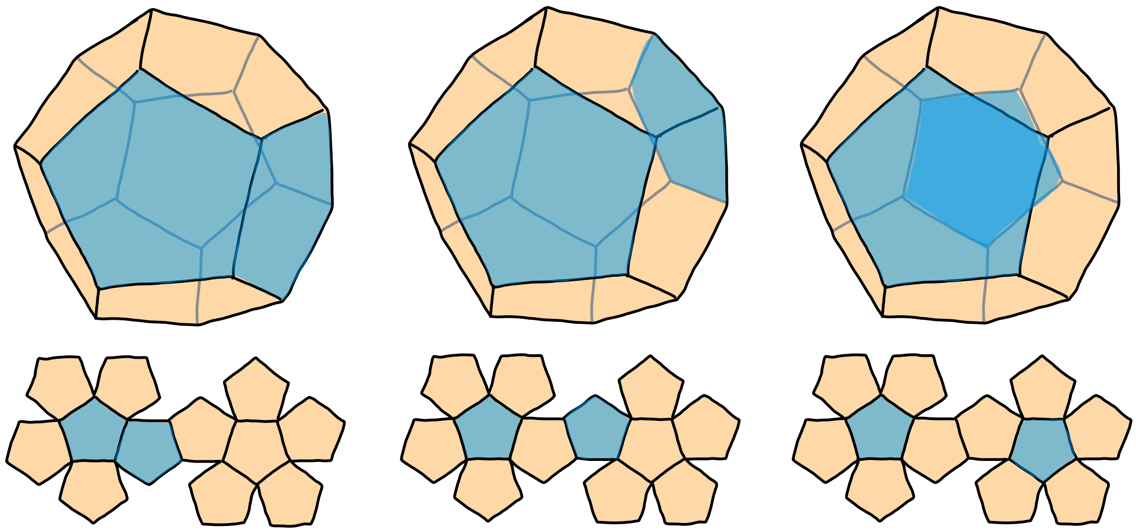


	1	15	20	24
	Identity	Edge ½ turns	Vertex ⅓ turns	Face turns
	$\frac{\cancel{12}^{6}\cdot 11}{\cancel{2}\cdot 1}$ $\frac{\cancel{12}^{2}\cdot 11\cdot 10}{\cancel{3}\cdot\cancel{2}\cdot 1}$	AA BB CC DD EE FF	AAA BBB CCC DDD	A CCCCC B DDDDD
k=2	$\binom{12}{2}=66$	6	0	1
k=3	$\binom{12}{3}=220$	0	4	0

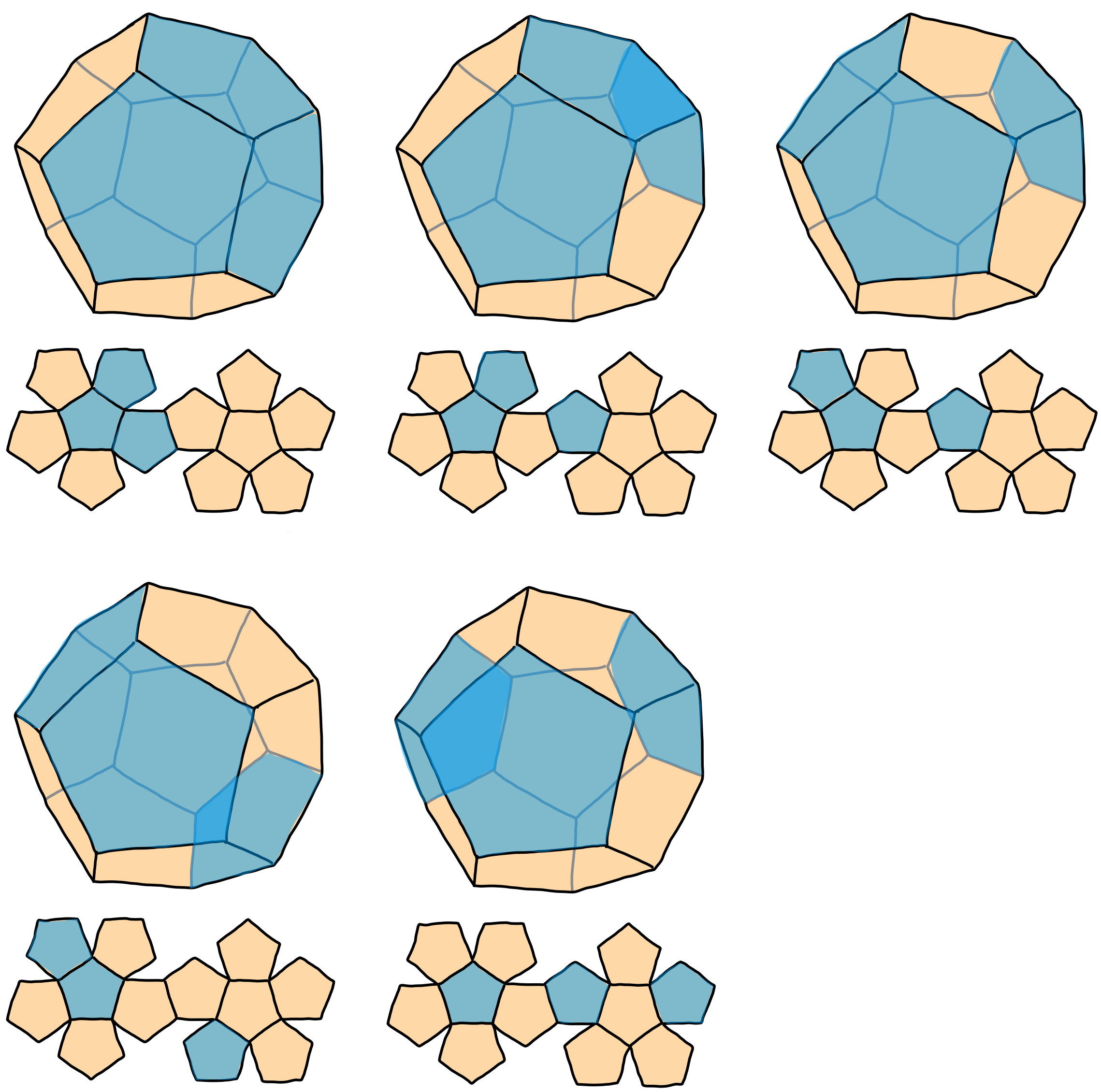
k=2 $\frac{1}{60}(66 + 15\cdot 6 + 20\cdot 0 + 24\cdot 1) = 180/60 = \boxed{3}$

k=3 $\frac{1}{60}(220 + 15\cdot 0 + 20\cdot 4 + 24\cdot 0) = 300/60 = \boxed{5}$

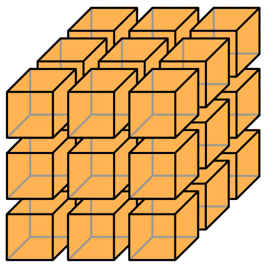
K=2 $\boxed{3}$

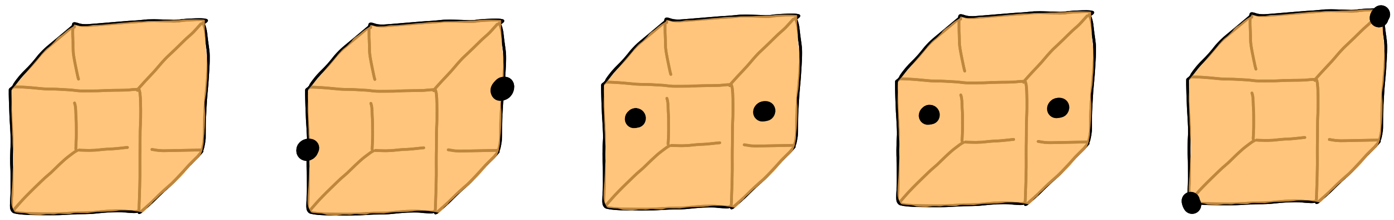


K=3 $\boxed{5}$

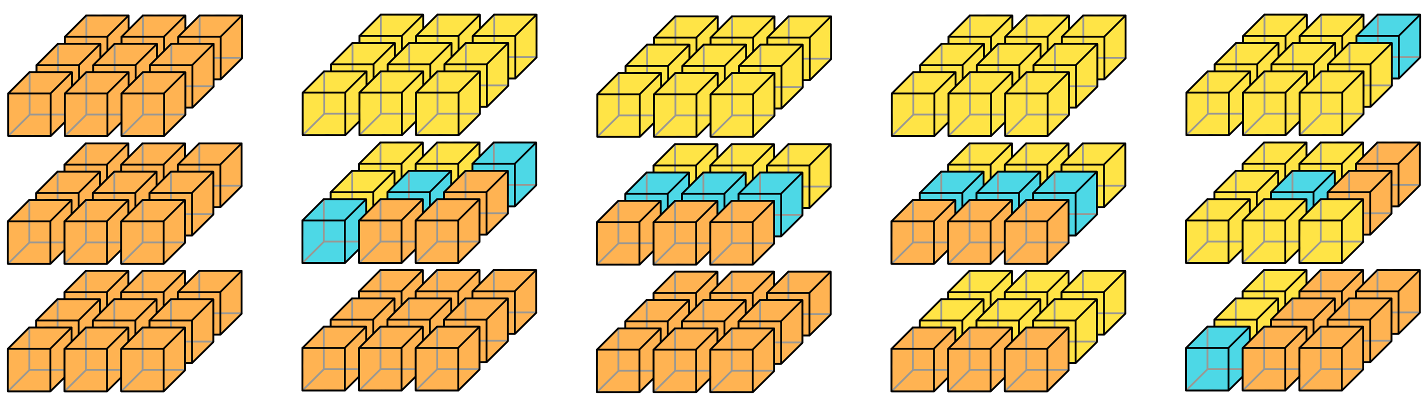


[5] How many ways can we choose two cubes from a $3 \times 3 \times 3$ array of 27 cubes, up to rotational symmetry? (This is not a *Rubik's Cube*. The symmetries are the 24 rotations we have studied of a solid cube.)





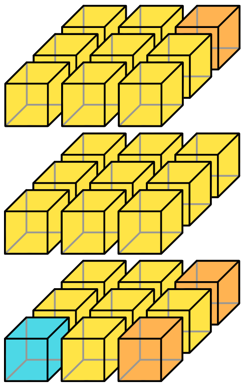
identity 1	½ turn	½ turn	¼ turn either way	⅓ turn either way
1	6	3	6	8



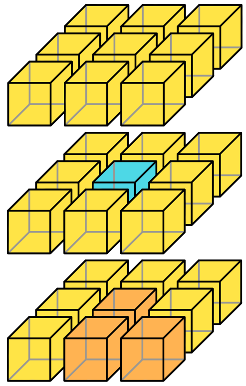
$\binom{27}{2} = \frac{27 \cdot 26}{2 \cdot 1} = 351$	3 cubes on axis 12 pairs $\binom{3}{2} + 12 = 15$	3 cubes on axis 12 pairs $\binom{3}{2} + 12 = 15$	3 cubes on axis 6 quads $\binom{3}{2} = 3$	3 cubes on axis 8 triplets $\binom{3}{2} = 3$

$\frac{1}{24}(351 + 9 \cdot 15 + 14 \cdot 3) = 528/24 = \boxed{22}$ ways to pick two cubes

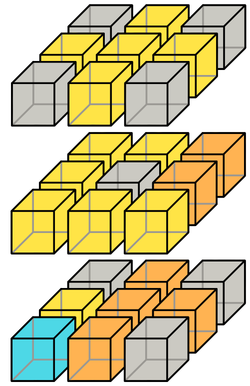
Check:



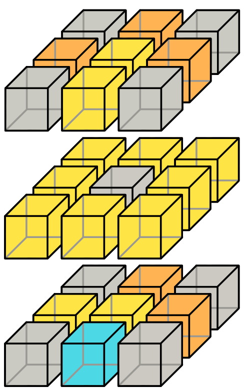
3 ways to choose two corners



3 ways to choose middle

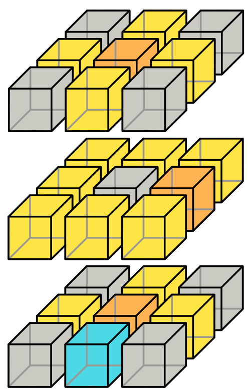


6 ways left to choose one corner

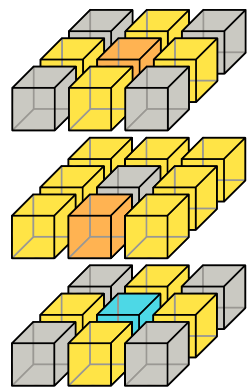


5 ways to choose two edges

(as we saw before)



3 ways to choose one edge, one face



2 ways to choose two faces

$$3+7+2+5+3+2 = \boxed{22}$$