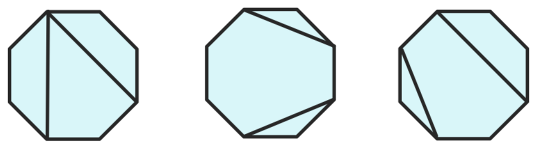
Final Exam

Combinatorics, Dave Bayer, April 20-23, 2021

To receive full credit for correct answers, please show all work.

[1] How many ways can we dissect an octagon using 2 cuts? Provide a check of your answer.
(You may solve the problem two different ways, or classify the possibilities, or draw every possibility.)



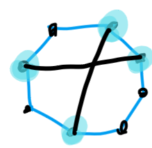
8 sides

$\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 28$ pairs of vertices

$\Rightarrow$ $28 - 8 = 20$ interior cuts

$\binom{20}{2} = \frac{20 \cdot 19}{2 \cdot 1} = 190$ pairs of cuts

$\binom{8}{4} = \frac{\cancel{8}^{2} \cdot 7 \cdot \cancel{6} \cdot 5}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 70$ crossing pairs



$\Rightarrow$ $190 - 70 = \boxed{120}$ 2 cut dissections

Check: There should be $2 \cdot 120 = 240$ ordered dissections

n	4	5	6	7	8
$\binom{n}{2}$	6	10	15	21	28
interior cuts	2	5	9	14	20

8 rotations
14 second cuts

112

8 rotations
$2+9=11$ second cuts

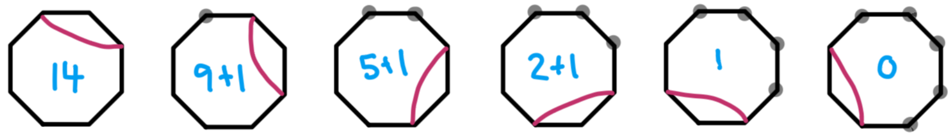
88

4 rotations
$5+5=10$ second cuts

40

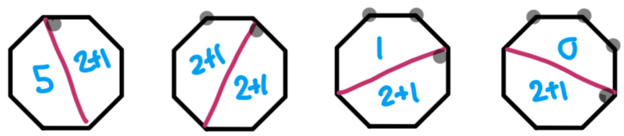
240 ✓

Check: Classify by position of first cut in sorting order:



$14+10+6+3+1+0=34$

$10+7+4+2+1=24$



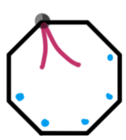
$8+6+4+3=21$

$8+7+6=21$

$10+10=20$

$34+24+21+21+20=120$ ☑

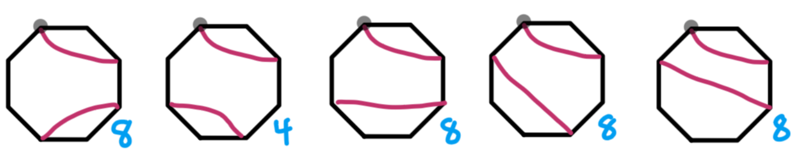
Check: Find up to rotational symmetry, and multiply by orbit sizes.



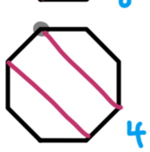
8 rotations

$\binom{5}{2}$ variants

80



36



$80+36+4=120$

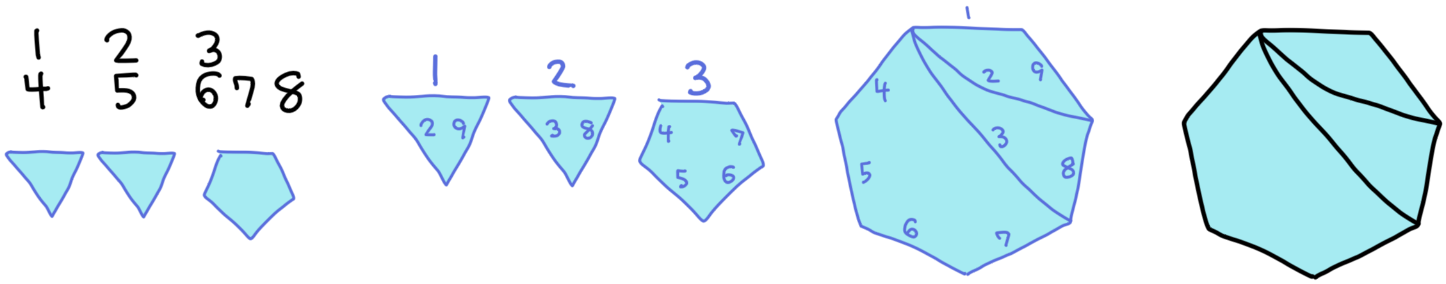
[2] For each of the following Young tableaux, find the dissection of an n-gon given by Stanley's correspondence.

1	2	3
4	5	6
7		
8		

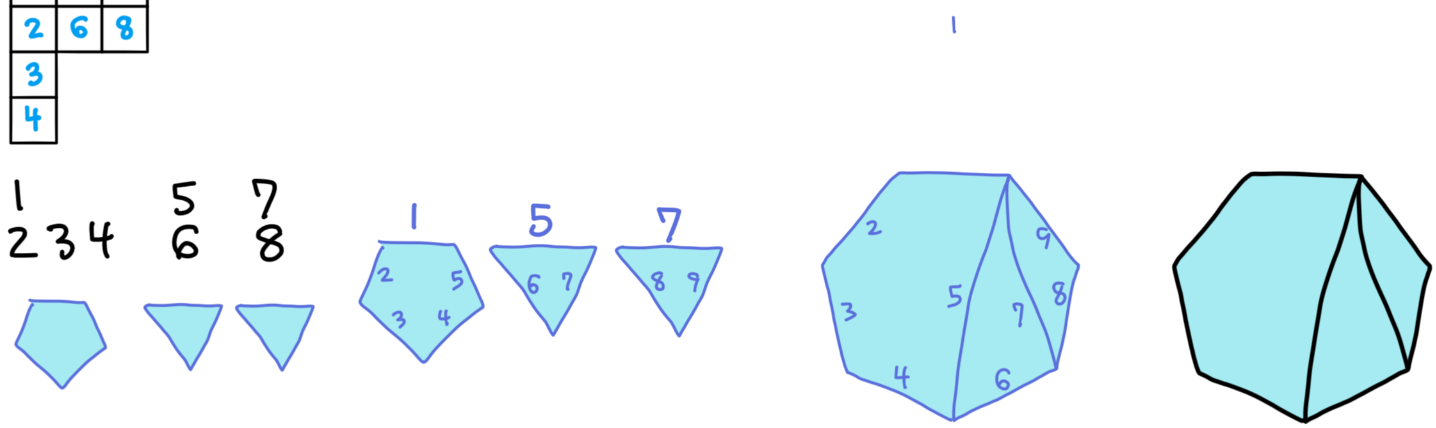
1	5	7
2	6	8
3		
4		

1	2	4
3	5	8
6		
7		

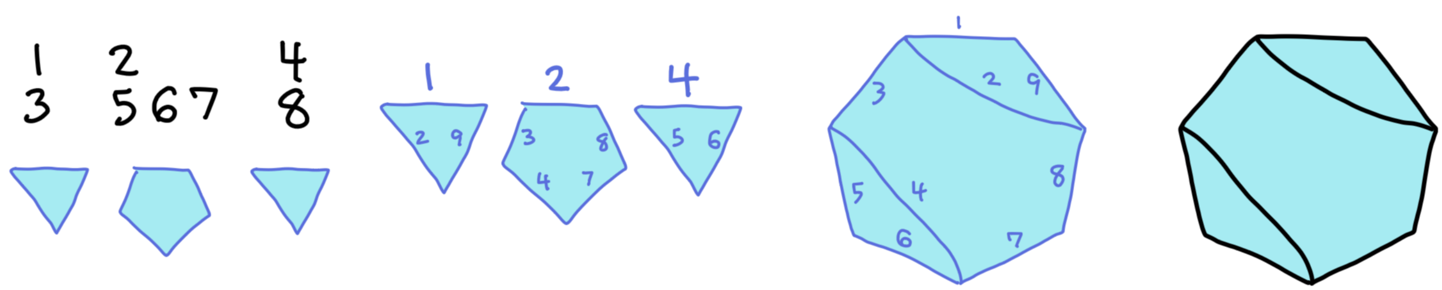
1	2	3
4	5	6
7		
8		



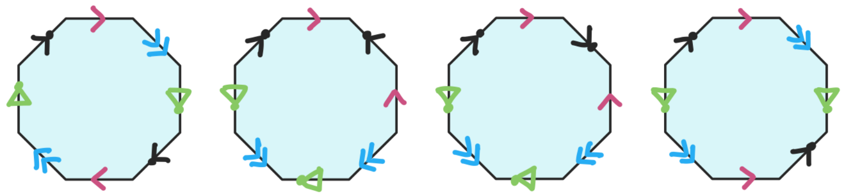
1	5	7
2	6	8
3		
4		

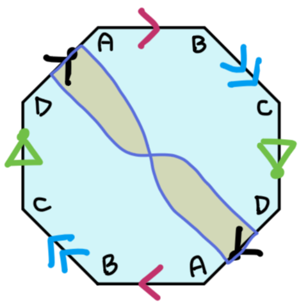


1	2	4
3	5	8
6		
7		



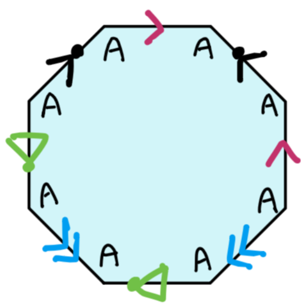
[3] Identify each of the following surfaces from their gluing diagrams, computing their Euler characteristic and deciding whether or not they are orientable. Which two surfaces are homeomorphic (topologically equivalent)?





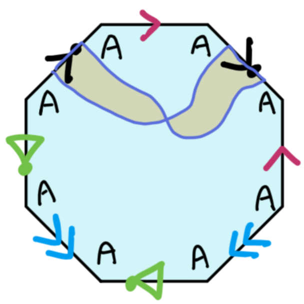
non-orientable

$\chi = 4 - 4 + 1 = 1$



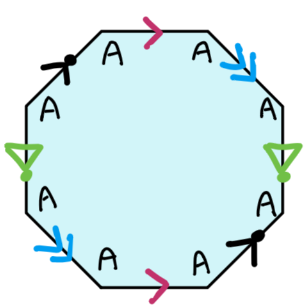
orientable

$\chi = 1 - 4 + 1 = -2$



non-orientable

$\chi = 1 - 4 + 1 = -2$

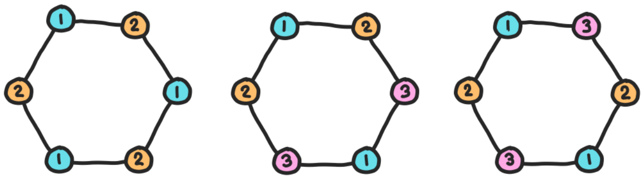


orientable

$\chi = 1 - 4 + 1 = -2$

Second and fourth surfaces are the same.

[4] How many ways can we properly color the vertices of a hexagon using n colors, up to rotational symmetry? Confirm your answer by drawing each of the possibilities for $n = 3$.
(For a proper coloring, adjacent vertices have distinct colors.)



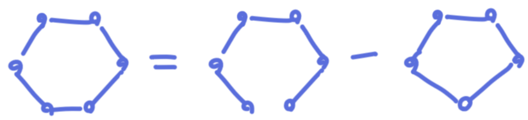
Let $f_K(n)$ be the chromatic polynomial of a K-cycle.

$$\boxed{f_K(n) = (n-1)^K + (-1)^K(n-1)}$$

Simplification of formula from class, we can prove by induction:

Basis: $K=2$  $f_2(n) = n(n-1) = (n-1)^2 + (n-1)$ ☑

Induction:



$$f_K(n) = n(n-1)^{K-1} - f_{K-1}(n)$$

$$(n-1)^K + (-1)^K(n-1) = n(n-1)^{K-1} - \left[(n-1)^{K-1} + (-1)^{K-1}(n-1)\right]$$ ☑

Apply Burnside's lemma $\frac{1}{|G|}\left[\#\text{fixed patterns, each } g \in G\right]$

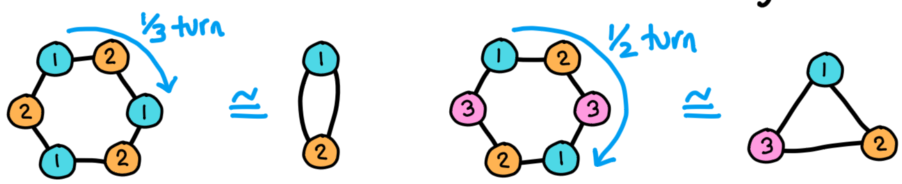
$G = \{0,1,2,3,4,5\}$ 6^{th} turns $|G| = 6$

0: $f_6(n) = (n-1)^6 + (n-1)$

1,5: no proper colorings fixed under $1/6^{th}$ turn

2,4: repeating pattern every $1/3$ turn $\iff K=2$ $f_2(n) = (n-1)^2 + (n-1)$

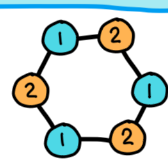
3: repeating pattern every $1/2$ turn $\iff K=3$ $f_3(n) = (n-1)^3 - (n-1)$



$$\boxed{f(n) = \frac{1}{6}\left[(n-1)^6 + (n-1)^3 + 2(n-1)^2 + 2(n-1)\right]}$$

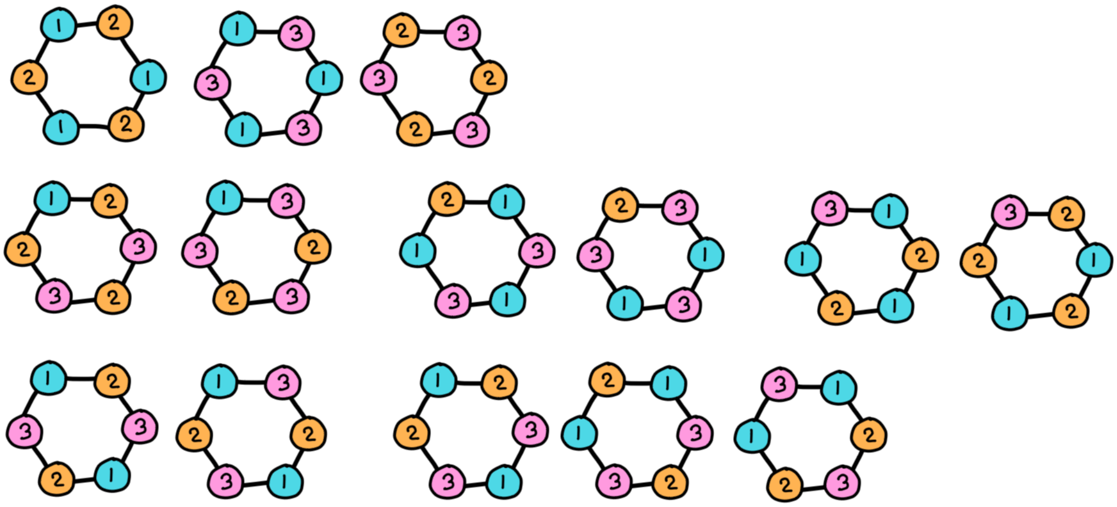
Check: $f(1) = 0$ ☑

$f(2) = 1$ ☑



n=2 only case,
up to symmetry

$$\boxed{f(3) = \tfrac{1}{6}\left[64 + 8 + 2\cdot 4 + 2\cdot 2\right] = 84/6 = 14}$$



What does $f(n)$ look like expanded out?

$$f(n) = \tfrac{1}{6}\left[(n-1)^6 + (n-1)^3 + 2(n-1)^2 + 2(n-1)\right]$$
$$= \tfrac{1}{6}\left[n^6 - 6n^5 + 15n^4 - 19n^3 + 14n^2 - 5n\right]$$

	n^6	n^5	n^4	n^3	n^2	n	1
$(n-1)^6$	1	-6	15	-20	15	-6	1
$(n-1)^3$				1	-3	3	-1
$2(n-1)^2$					2	-4	2
$2(n-1)$						2	-2
	1	-6	15	-19	14	-5	0

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Check: $f(1) = 0$ ☑ 30-30
$f(2) = 1$ ☑

64		1						
$-6\cdot 32$	-1	-1	0					
$+15\cdot 16$	1	1	1	1				
$-19\cdot 8$	-1	0	0	-1	-1			
$+14\cdot 4$			1	1	1	0		
$-5\cdot 2$					-1	0	-1	
	0	0	0	1	-1	0	-1	0
					1	0	-1	0
6						1	1	0 ☑

Check in binary
(for computer scientists,
or anyone bored by conventional
arithmetic)

```
In[1]:= Try[f_, d_] := Module[{},
          Print[f /. m → n - 1 // Expand];
          Print[f /. n → m + 1 // Expand];
          Print[Table[f / d /. n → k /. m → k - 1, {k, 5}]];]

In[2]:= Try[m^6 + m^3 + 2 m^2 + 2 m, 6]
```

$-5\,n+14\,n^2-19\,n^3+15\,n^4-6\,n^5+n^6$

$2\,m+2\,m^2+m^3+m^6$

{0, 1, 14, 130, 700}

```
In[3]:= Try[n m (m^4 - m^3 + m^2 + 2), 6]
```

$-5\,n+14\,n^2-19\,n^3+15\,n^4-6\,n^5+n^6$

$2\,m+2\,m^2+m^3+m^6$

{0, 1, 14, 130, 700}

```
In[4]:= Try[n m^5 - n m (m^3 - m^2 + m - 1) + n m (n - 2) + 2 n m, 6]
```

$-5\,n+14\,n^2-19\,n^3+15\,n^4-6\,n^5+n^6$

$2\,m+2\,m^2+m^3+m^6$

{0, 1, 14, 130, 700}

```
In[5]:= Try[m^6 + m + 2 n m + n m (n - 2), 6]
```

$-5\,n+14\,n^2-19\,n^3+15\,n^4-6\,n^5+n^6$

$2\,m+2\,m^2+m^3+m^6$

{0, 1, 14, 130, 700}

```
In[6]:= Try[(n^2 - n + 1) n m (n^2 - 4 n + 5), 6]
```

$-5\,n+14\,n^2-19\,n^3+15\,n^4-6\,n^5+n^6$

$2\,m+2\,m^2+m^3+m^6$

{0, 1, 14, 130, 700}

```
In[7]:= Try[n (n - 1) ((n - 1)^4 - (n - 1)^3 + (n - 1)^2 - (n - 2)), 1]
```

$-5\,n+15\,n^2-20\,n^3+15\,n^4-6\,n^5+n^6$

$m+m^6$

{0, 2, 66, 732, 4100}

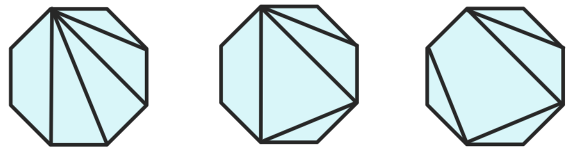
```
In[8]:= Try[2 m^3 - m, 1]
```

$-1+5\,n-6\,n^2+2\,n^3$

$-m+2\,m^3$

{0, 1, 14, 51, 124}

[5] How many ways can we dissect an octagon using 4 cuts, up to dihedral (rotations and flips) symmetry? Confirm your answer by drawing each of the possibilities. Which patterns are not chiral?



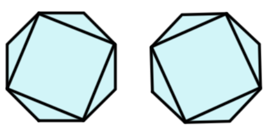
Dihedral group G, $|G| = 16$ 8 rotations, 8 flips

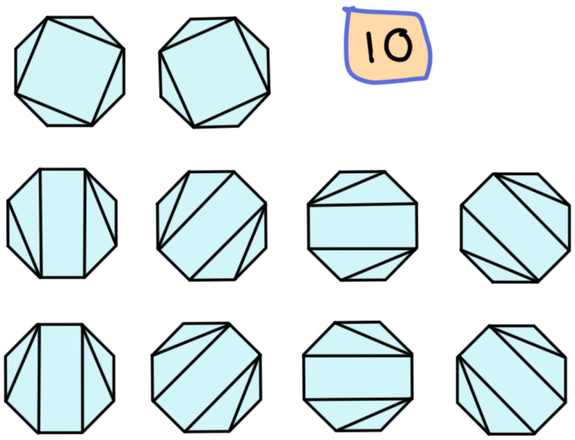
8^{th} turn rotations:

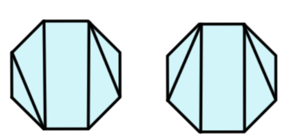
0: All raw dissections $\frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$

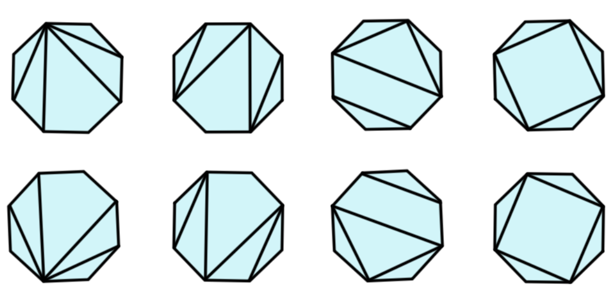
$n=8$, $k=4$: $\frac{1}{5}\binom{5}{4}\binom{11}{4} = \frac{11\cdot 10\cdot \cancel{9}^{3}\cdot \cancel{8}}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = \boxed{330}$

1,3,5,7: No dissections fixed by eighth turn

2,6:  $\boxed{2}$

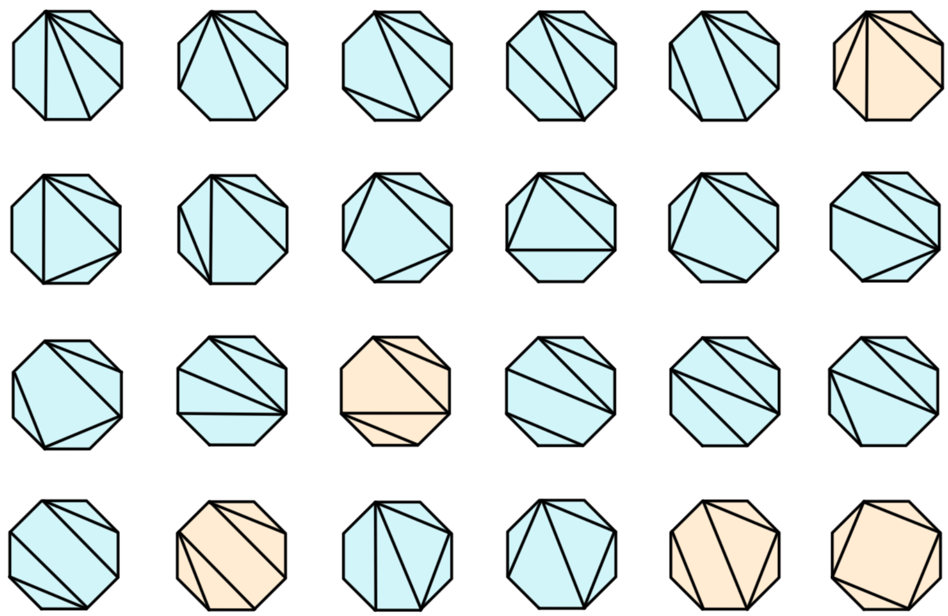
4:  $\boxed{10}$

 :  $\boxed{2}$

 :  $\boxed{8}$

Burnside's formula:

$$\frac{1}{16}\left[330+\underbrace{2+2+10+2+2+2+2}_{22}+\underbrace{8+8+8+8}_{32}\right] = \frac{384}{16} = \boxed{24}$$



The following 5 dissections are not chiral: