Combinatorics
January 12, 2021

Feller $\sum$ $\int$

Aigner

How many words length 2

#1 letter	#2 letter
A	x
B	y
C	z

	x	y	z
A	AX	AY	AZ
B	BX	BY	BZ
C	CX	CY	CZ

$3 * 3 = 9$

Binomial Coefficients

$\binom{n}{k}$ "n choose k" = # number of subsets of size k of n things.

$6 = \binom{4}{2}$

A	B	C	D
X	X		
X		X	
X			X
	X	X	
	X		X
		X	X

Pascal's Triangle

1
1 1
1 2 + 1 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\binom{0}{0}$
$\binom{1}{0}$ $\binom{1}{1}$
$\binom{2}{0}$ $\binom{2}{1} + \binom{2}{2}$
$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

divide and conquer

$\binom{4}{2}=6$

1 2 3 4

pick 2

use 1?

yes → $1 \ldots \binom{3}{1}$ 12 13 14

no → $\binom{3}{2}$ 23 24 34

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\boxed{\binom{n+1}{k} - \binom{n}{k}} = \binom{n}{k-1}$$

$f(x) : \mathbb{R} \to \mathbb{R}$

$g(n) : \mathbb{Z} \to \mathbb{Z}$

$$\boxed{f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}}$$

$$\boxed{\Delta g(n) = g(n+1) - g(n)}$$

$$g(n) = \binom{n}{k}$$

$$\Delta g(n) = \boxed{\binom{n+1}{k} - \binom{n}{k}} = \binom{n}{k-1}$$

$$\frac{d}{dx} e^x = e^x \qquad \Delta 2^n = 2^{n+1} - 2^n = 2^n$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}$$

$\times 0$ ← $1, x, x^2, x^3, \ldots$

$\times 1 \quad \times 2 \quad \times 3$

$$\underbrace{1 + \underbrace{2 + 3 + 4} + 5} = 15$$

$$\underbrace{1 + \underbrace{2 + 3} + 4} = 10$$

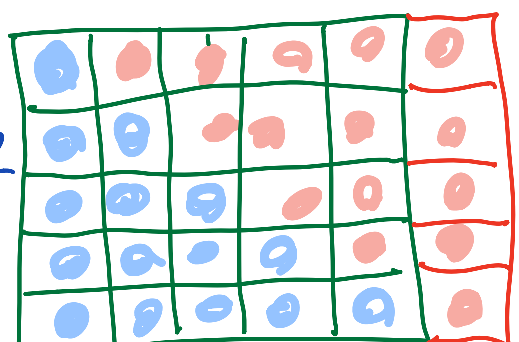
1

$+2$

$+3$

$+4$

$+5$



5 +

4 +

3 +

2 +

1

$$5 \times 6 / 2 = 15$$

$$n(n+1)/2$$

$$\sum_{i=1}^{n} i^m \quad \text{for any power} \qquad m=1$$

$$g(n) = 1+2+\dots+n$$

$$\Delta g(n) = g(n+1)-g(n) = n+1 = \binom{n}{1}+\binom{n}{0}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\Delta g(n) = \binom{n}{1}+\binom{n}{0}$$

$$\Rightarrow g(n) = \binom{n}{2}+\binom{n}{1}+\cancel{0}$$

$$= \frac{n(n-1)}{2}+\frac{2n}{2} = \frac{n^2+n}{2}$$

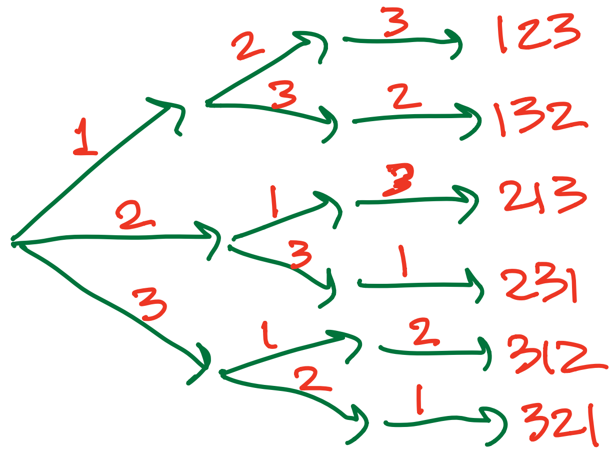
$$= \frac{n(n+1)}{2}$$

$$\binom{n}{k} \qquad \text{overcounting}$$

$n! =$ all permutations of $1\dots n$ (or any n things)

$$= n\cdot(n-1)\cdots 3\cdot 2\cdot 1$$

$n!$



$3! =$
$3\cdot 2\cdot 1 = 6$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2\cdot 2} = 6$$

keep | discard

ignore order

12	13	14
12|34, 21|34	13|24, 31|24	14|23, 41|23
12|43, 21|43	13|42, 31|42	14|32, 41|32
23	**24**	**34**
23|14, 32|14	24|13, 42|13	34|12, 43|12
23|41, 32|41	24|31, 42|31	34|21, 43|21

$4!/2!2!$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\overbrace{\underbrace{1\ 2\ 3\ 4}_{k \text{ keep}} \,\Big|\, \underbrace{5\ 6\ 7\ 8\ 9}_{n-k \text{ discard}}}^{n}$$

$$\binom{8}{3} = \frac{8\cdot 7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{3\cdot 2\cdot 1\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$$

$$= \frac{\overset{4}{\cancel{8}}\cdot 7\cdot \overset{2}{\cancel{6}}}{\cancel{3}\cdot \cancel{2}\cdot 1} = 56$$

"bars & stars" argument

how many monomials of deg d in n variables?
terms

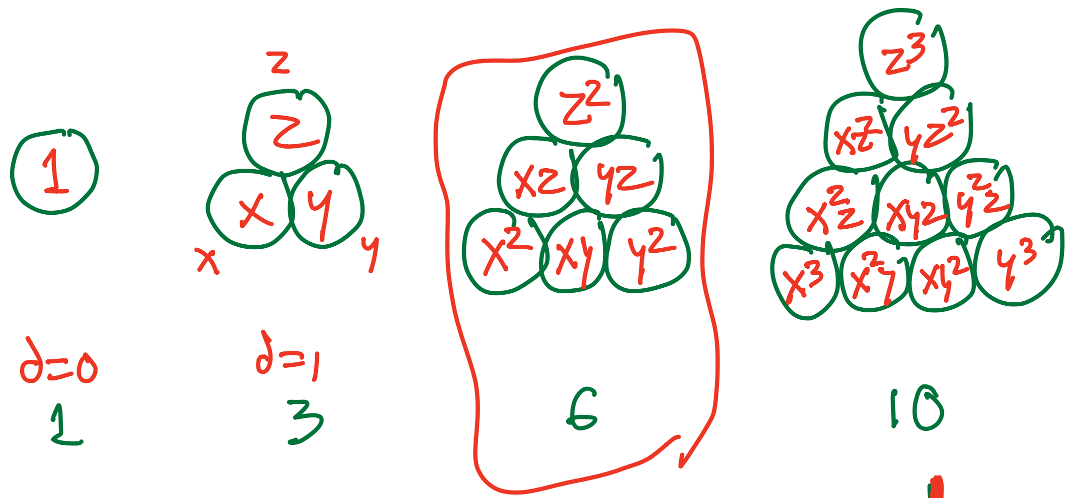
$n=3 \quad x, y, z$

$d=0 \quad x^0y^0z^0 = 1$

$d=1 \quad x, y, z$

$d=2 \quad x^2, y^2, z^2, xy, xz, yz$

$d=3$ $\quad x^3, y^3, z^3, x^2y, x^2y, xy^2, xz^2, yz^2, y^2z, xyz$



Pascal's Triangle

1

1 1

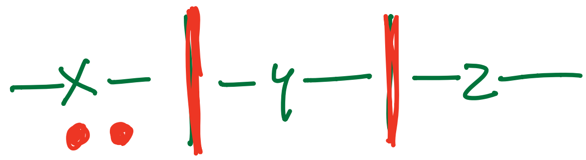
1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{4}{2}=6$



x^2	••||
xy	•|•|
xz	•||•
y^2	|••|
yz	|•|•
z^2	||••

deg 3 in x, y, z $\quad$ 2 dividers $\quad x|y|z$

3 balls $\quad$ •••

••|•| $= x^2y$

x y z

$\binom{5}{2} = \frac{5\cdot \cancel{4}^{2}}{\cancel{2}\cdot 1} = 10$

xyz $\quad$ •|•|•