# Discrete Calculus

$$F'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

discrete $\varepsilon = 1$

$$\Delta F(x) = f(x+1) - f(x)$$

replace $x$ by $n$

$$\Delta f(n) = f(n+1) - f(n)$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$0! = 1$

$\binom{n}{0} = 1$ $\quad x^0 \quad n^0$

$\binom{n}{1} = \frac{n}{1} = n$ $\quad x^1 \quad n^1$

$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2 - n}{2}$ like $x^2$ not quite

---

$\binom{n}{k} =$ $1, 2, 3, \ldots n$ choose $k$ of them

? use 1?

Y $\to$ 1... $\quad 1, \underbrace{2, 3, \ldots n}_{n-1}$ left $\underbrace{\text{chose } k-1}_{k-1}$ $\quad \binom{n-1}{k-1}$

N $\to$ ... $\quad 2, 3, \ldots n$ left choose $k$ $\quad \binom{n-1}{k}$

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$$

same thing

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\underbrace{\boxed{\binom{n+1}{k} - \binom{n}{k}}} = \boxed{\binom{n}{k-1}}$$

by definition

if $\boxed{f(n) = \binom{n}{k}}$ then $\Delta f(n) = f(n+1) - f(n)$

$$= \boxed{\binom{n+1}{k} - \binom{n}{k}}$$

$$\boxed{\Delta f(n) = \boxed{\binom{n}{k-1}}}$$

$$\binom{n}{4} \xrightarrow{\Delta} \binom{n}{3} \xrightarrow{\Delta} \binom{n}{2} \xrightarrow{\Delta} \binom{n}{1} \xrightarrow{\Delta} \binom{n}{0} \xrightarrow{\Delta} 0$$

$$x^4 \xrightarrow{\frac{1}{4}\frac{d}{dx}} x^3 \xrightarrow{\frac{1}{3}\frac{d}{dx}} x^2 \xrightarrow{\frac{1}{2}\frac{d}{dx}} x \xrightarrow{\frac{d}{dx}} 1 \xrightarrow{\frac{d}{dx}} 0$$

---

application: $\boxed{g(n) = 1+2+\dots+n}$

$$\Rightarrow \Delta g(n) = \begin{array}{c} g(n+1) \\ -g(n) \end{array} = \begin{array}{l} \cancel{1+2+\dots+n}+n+1 \\ -(\cancel{1+2+\dots+n}) \end{array}$$

$$= n+1$$

discrete "integration" problem

what $g(n)$ has discrete derivative $n+1$ ?

$$n+1 = \binom{n}{1} + \binom{n}{0}$$

"  "  "

$$\binom{n}{0} = 1$$
$$\binom{n}{1} = n$$
$$\binom{n}{2} = \frac{n(n-1)}{2}$$

want

$$\boxed{g(n) = \binom{n}{2} + \binom{n}{1} + C}$$

$$\downarrow \Delta \qquad \downarrow \Delta \qquad \downarrow \Delta \qquad \downarrow \Delta$$

$$\boxed{\Delta g(n) = \binom{n}{1} + \binom{n}{0}} \qquad 0$$

have

---

$$g(n) = \binom{n}{2} + \binom{n}{1} + C$$
$$1+2 = g(2) = 1 + 2 + 0$$

figure out $C=0$

---

$$g(n) = \underbrace{\binom{n}{2} + \binom{n}{1}}_{} = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

$$= \boxed{\binom{n+1}{2}}$$

apply this

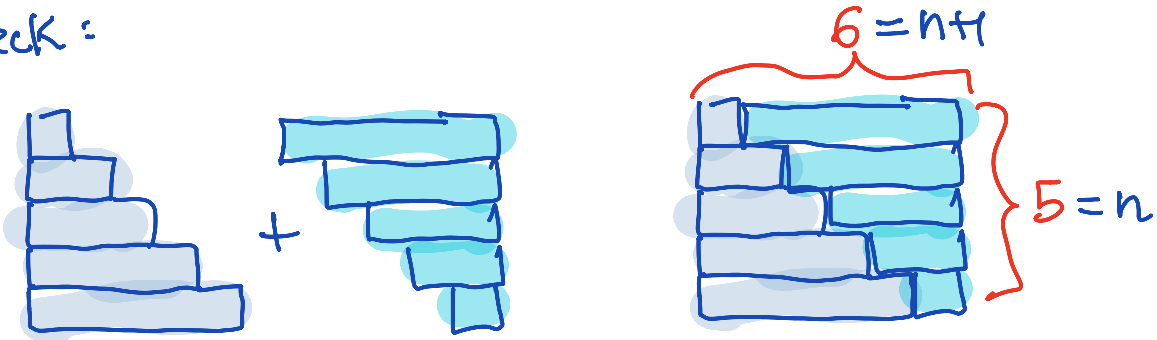
$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$$

use $n$ for $n-1$
use 2 for $k$

nice form of answer:

$$\boxed{1+2+\ldots n = g(n) = \binom{n+1}{2} = \frac{(n+1)n}{2}}$$

check:




$1+2+3+4+5$    $1+2+3+4+5$    $2(1+2+3+4+5) = 6\cdot 5$

$(n=5)$

$$\boxed{1+2+3+4+5 = \frac{6\cdot 5}{2}}$$

---

$$h(n) = 1^2+2^2+3^2+\ldots+n^2$$

$$\Delta h(n) = \underbrace{h(n+1)}_{\cancel{1^2+\ldots+n^2}+(n+1)^2} - \underbrace{h(n)}_{\cancel{1^2+\ldots+n^2}} = (n+1)^2 = n^2+2n+1$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$2\binom{n}{2} = n(n-1) = n^2-n$$

---

$$\underbrace{(n^2+2n+1)}_{\Delta h(n)} - \underbrace{(n^2-n)}_{2\binom{n}{2}} = \underbrace{3n}_{3\binom{n}{1}} + \underbrace{1}_{\binom{n}{0}}$$

$$\boxed{\Delta h(n) = 2\binom{n}{2} + 3\binom{n}{1} + \binom{n}{0}}$$

check: $\Delta h(2) = 2\binom{2}{2} + 3\binom{2}{1} + \binom{2}{0} = 2+6+1 = 9$

$\Delta h(2) = (2+1)^2 = 9$ ✓

$$h(n) = 2\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1} + \cancel{c} = 0 \qquad h(1) = 1^2 = 1$$

?

$\Delta\downarrow$   $\Delta\downarrow$   $\Delta\downarrow$   $\Delta\downarrow$   $\Delta\searrow 0$

$$\boxed{\Delta h(n) = 2\binom{n}{2} + 3\binom{n}{1} + \binom{n}{0}}$$

answer

$$\boxed{h(n) = 2\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1}}$$

$$= 1^2 + 2^2 + \dots + n^2$$

$$\frac{2n(n-1)(n-2)}{6} + \frac{3n(n-1)}{2} + n$$

check:
$n=4$

20

$$1+4+9+16 = 30$$

10

$$\underbrace{2\binom{4}{3}}+3\binom{4}{2}+\binom{4}{1}$$

$$2\cdot 4 + 3\cdot 6 + 4$$

$$8 + 18 + 4 = 30 \checkmark$$

---

| 12 | 13 | 14 |
|---|---|---|
| 12\|34, 21\|34, 12\|43, 21\|43 | 13\|24, 31\|24, 13\|42, 31\|42 | 14\|23, 41\|23, 14\|32, 41\|32 |
| **23** | **24** | **34** |
| 23\|14, 32\|14, 23\|41, 32\|41 | 24\|13, 42\|13, 24\|31, 42\|31 | 34\|12, 43\|12, 34\|21, 43\|21 |

want formula for $\binom{4}{2}$

have formula for all permutations $4! = 4\cdot3\cdot2\cdot1$

n/e given a permutation, grab first two elements as our subset

$$3214 \Rightarrow \boxed{32|14} \rightarrow \{2,3\}$$

ignore order

ignore completely

How many permutations give $\{2,3\}$ as subset?

23

| | |
|---|---|
| 23\|14 | 32\|14 |
| 23\|41 | 32\|41 |

(2! orders first part)(2! orders 2nd part)

$\binom{7}{3}$    $\boxed{4\ 1\ 6}\boxed{2\ 5\ 3\ 7} \longrightarrow \{1,4,6\}$

$$\frac{(7!\ \text{permutations})}{\underbrace{(\text{divide by } 3!)}\ \underbrace{(\text{divide } 4!)}}$$

$$\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7\cdot\cancel{6}\cdot5\cdot4\cdot3\cdot2\cdot1}{\cancel{3\cdot2}\cdot1\cdot4\cdot3\cdot2\cdot1} = 35$$

| | | | | |
|---|---|---|---|---|
| 1 2 3 | 1 3 5 | 1 5 6 | 2 3 7 | 2 6 7 |
| 1 2 4 | 1 3 6 | 1 5 7 | 2 4 5 | 3 4 5 |
| 1 2 5 | 1 3 7 | 1 6 7 | 2 4 6 | 3 4 6 |
| 1 2 6 | 1 4 5 | 2 3 4 | 2 4 7 | 3 4 7 |
| 1 2 7 | 1 4 6 | 2 3 5 | 2 5 6 | 3 5 6 |
| 1 3 4 | 1 4 7 | 2 3 6 | 2 5 7 | 3 5 7 |
| 3 6 7 | 4 5 6 | 4 5 7 | 4 6 7 | 5 6 7 |