March 25

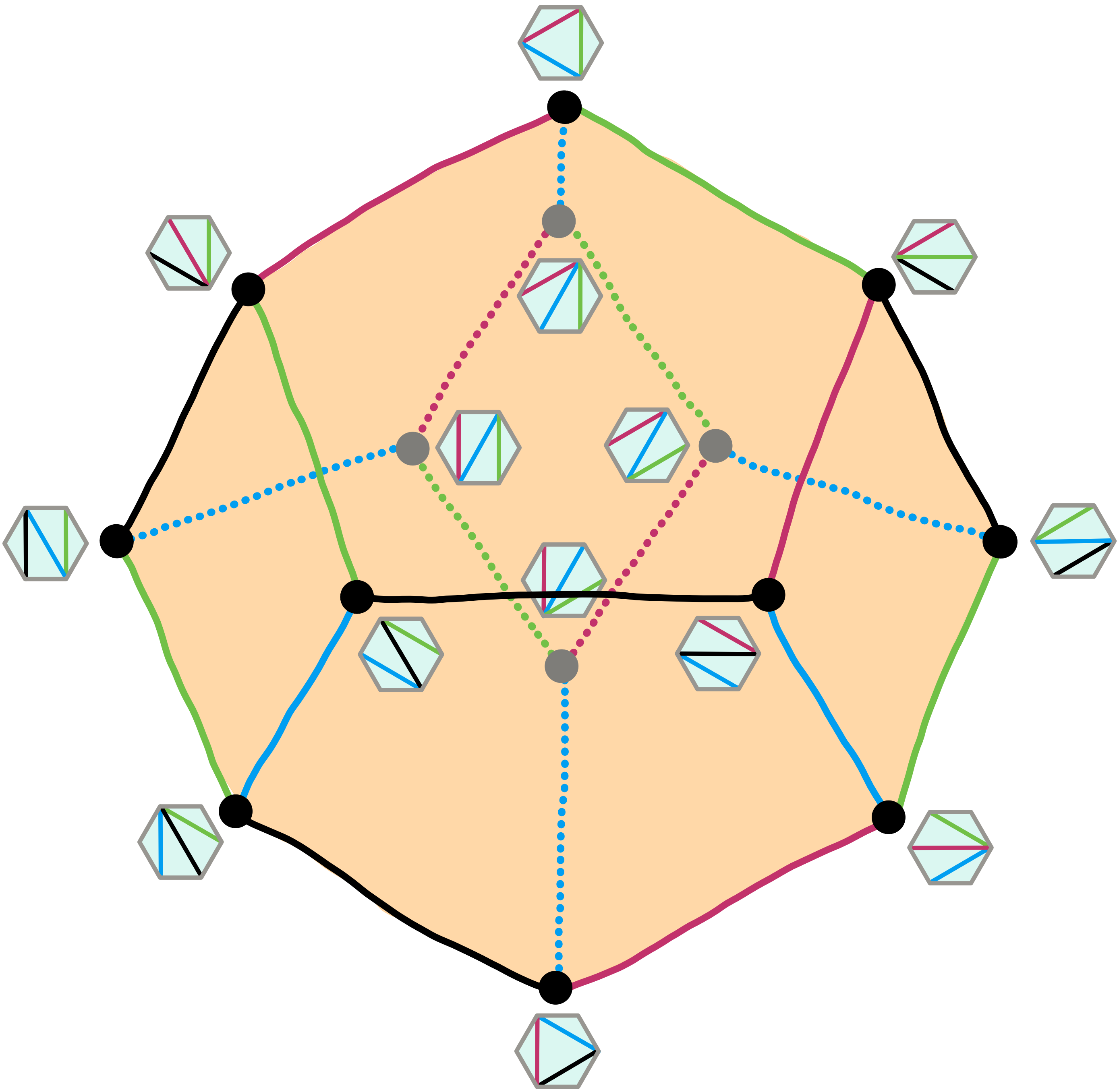
The Associahedron in $\mathbb{R}^3$

Euler characteristic of boundary

$\chi = v - e + f = 14 - 21 + 9 = 2$

like any sphere

c	f	e	v
1	9	21	14



$T(n,k)$ = number of dissections of an n-gon by k cuts

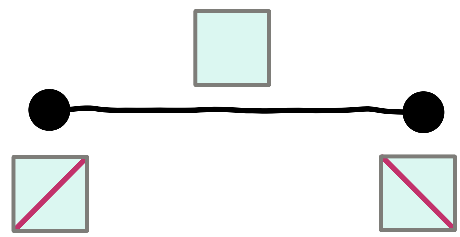
n-gon \ k cuts	0	1	2	3	4	5	6
3	1						
4	1	2					
5	1	5	5				
6	1	9	21	14			
7	1	14	56	84	42		
8	1	20	120	300	330	132	
9	1	27	225	825	1485	1287	429

⇐ the associahedron in $\mathbb{R}^3$

= Catalan numbers

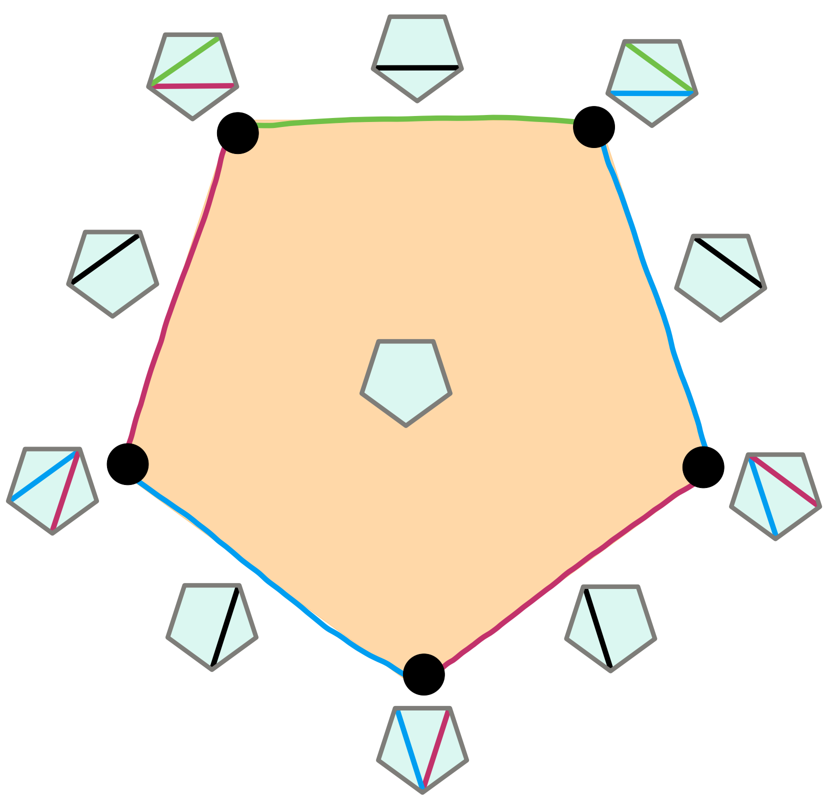
the associahedron in $\mathbb{R}^1$:

1	2

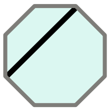


the associahedron in $\mathbb{R}^2$:

1	5	5



1 cut:

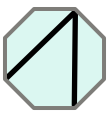


n	①
3	
4	2
5	5
6	9
7	14
8	20
9	27

n	$\binom{n}{2}$	$-n$	①
4	6	4	2
5	10	5	5
6	15	6	9
7	21	7	14
8	28	8	20
9	36	9	27

$\binom{n}{2}$ pairs of vertices
$-n$ sides of n-gon
$=$ # interior edges

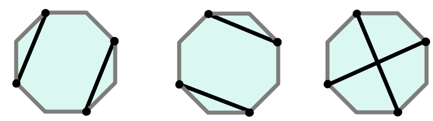
2 cuts:



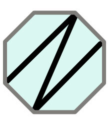
n	②
3	
4	
5	5
6	21
7	56
8	120
9	225

n	①	$\binom{①}{2}$	$-\binom{n}{4}$	②
5	5	10	5	5
6	9	36	15	21
7	14	91	35	56
8	20	190	70	120
9	27	351	126	225

pairs of interior edges
− crossing pairs

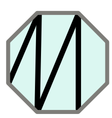


3 cuts:



n	③
3	
4	
5	
6	14
7	84
8	300
9	825

4 cuts:



n	④
3	
4	
5	
6	
7	42
8	330
9	1485

Can be done ad hoc.
Gets harder...

Many approaches

Formula:

$T(n,k) =$ number of dissections of an n-gon by k cuts

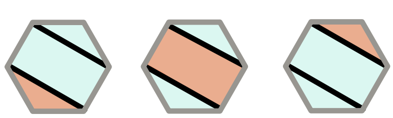
$$= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$$

1890 Cayley
... 2000 Przytycki, Sikora

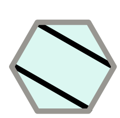
Meaning of each part:

$\boxed{\frac{1}{k+1}}$

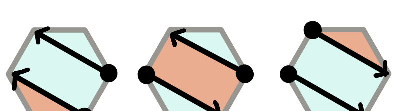
We overcount, then divide. k cuts ⇒ k+1 regions.
Count k cuts with a marked region.



counts



3 times

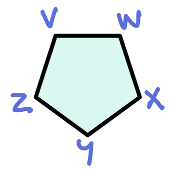


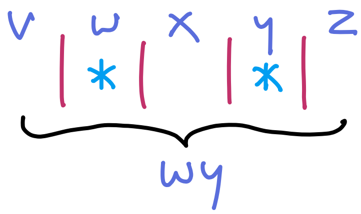
Orient each cut to keep marked region on the left.
Each cut now has a "start" •

$\boxed{\binom{n+k-1}{k}}$

This looks like a "bars & stars" monomial count.

The k cuts can start anywhere, including several from the same vertex.

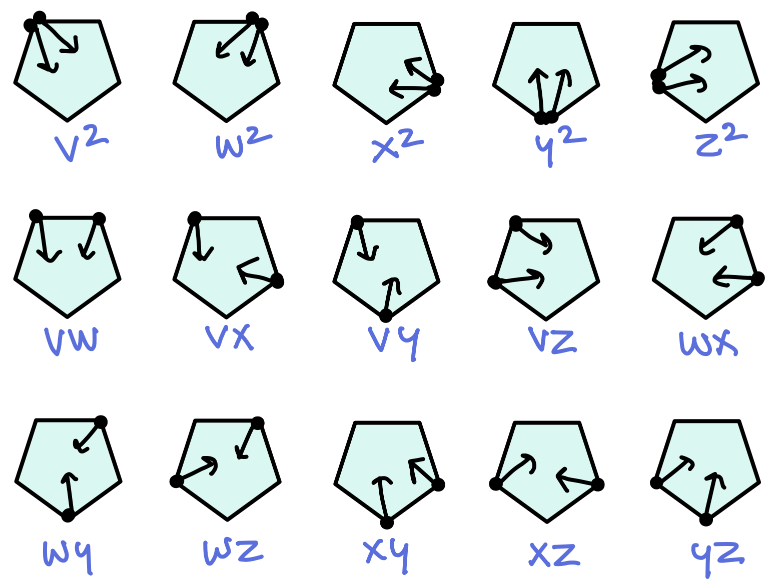




n−1	k
|	*

$\binom{n+k-1}{k}$

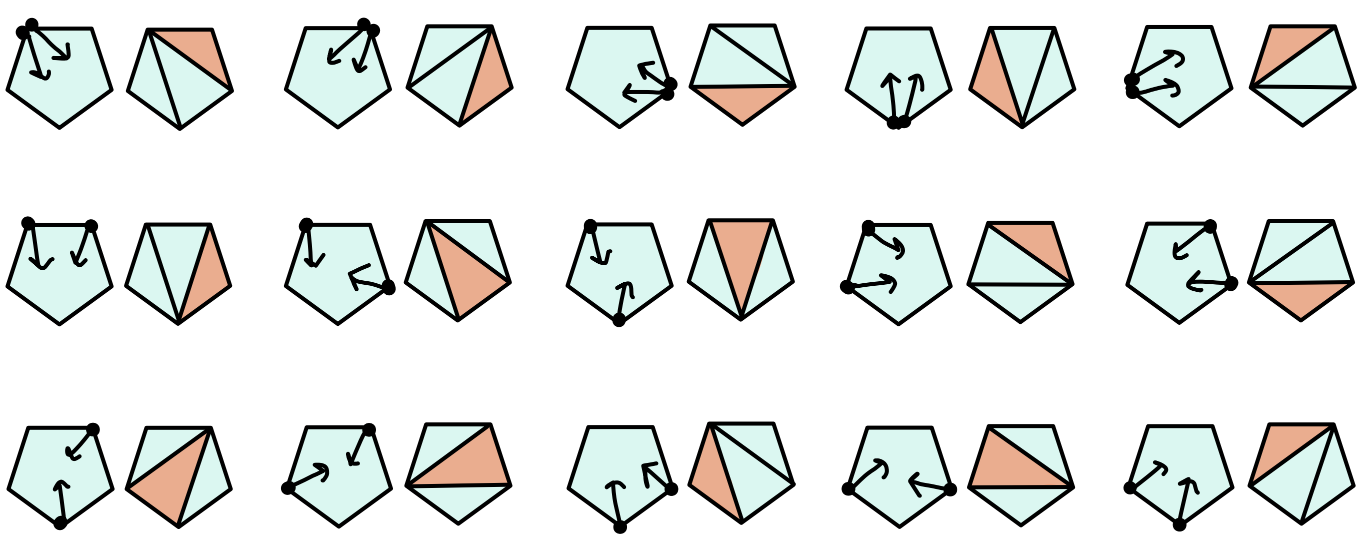
Choose the k *'s



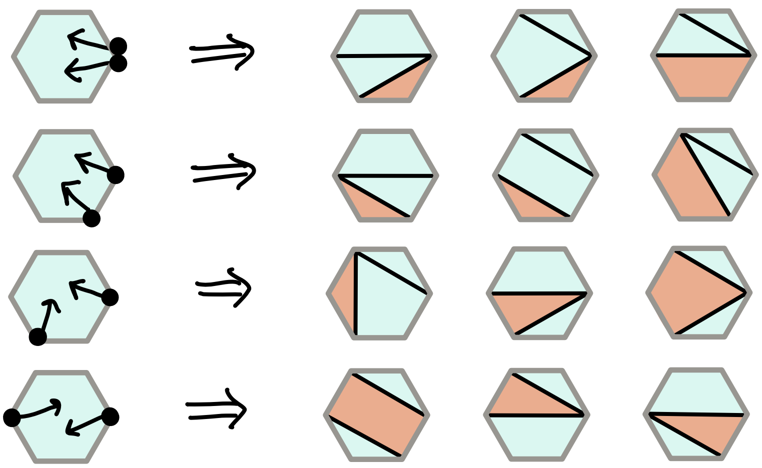
$\boxed{\binom{n-3}{k}}$

Counts ways to finish diagram, so cut directions are compatible with a choice of marked region.

$n=5,\ k=2 \Rightarrow \binom{2}{2}=1$, unique way for our example.



$\boxed{\binom{n-3}{k}}$ $n=6,\ k=2 \Rightarrow \binom{3}{2}=3$



Anyone up for a real life bonus question?

Challenge: Without looking at 2000 Przytycki, Sikora, find your own proof of this last step.
Play with this. I believe there could be a simpler argument.

Apparently unrelated topic (of course they're related!)

Young tableaux

How many ways can we grow a staircase shape, step by step?

$1 = 1$

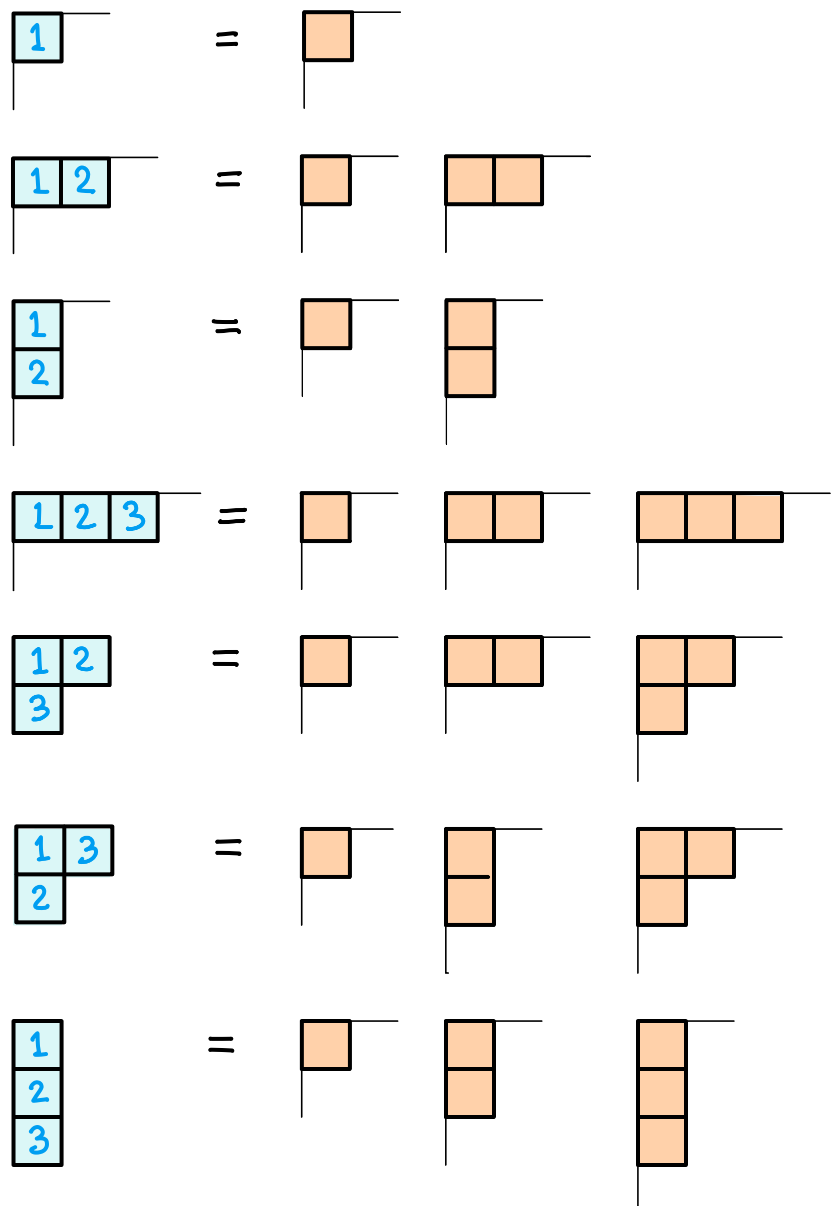
$2 = 2$

$2 = 1+1$

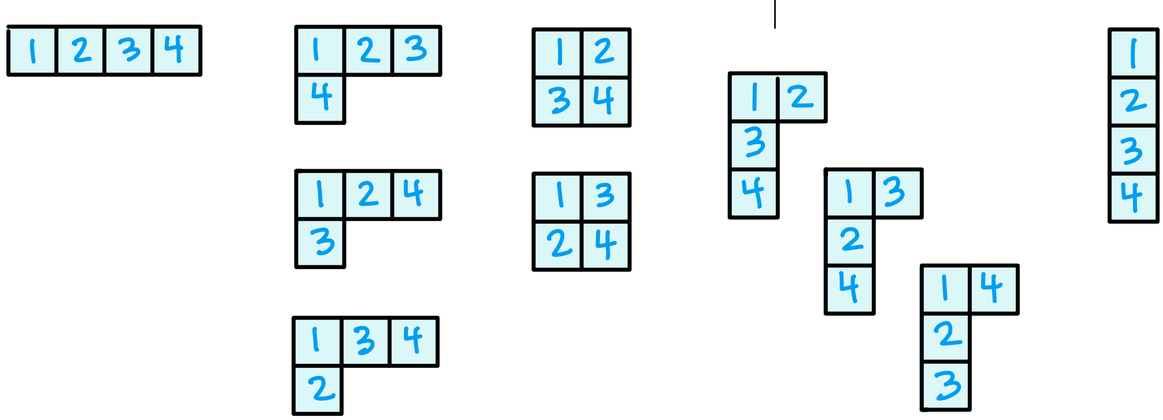
$3 = 3$

$3 = 2+1$

$3 = 1+1+1$



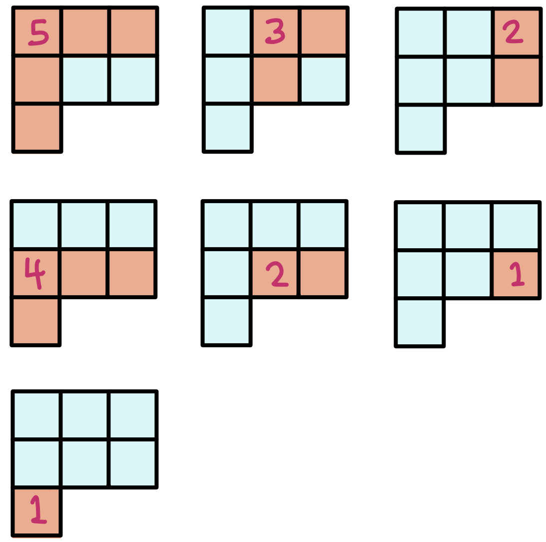
4:



Hook length formula

For each cell, record the length of the "hook" down or over.

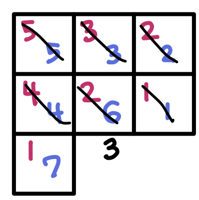
5	3	2
4	2	1
1		



For n cells, divide n! by the product of the hook lengths.

$$\frac{7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{5}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{4}\cdot\cancel{2}\cdot 1\cdot 1} = 21$$

or



= 21

$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 7\\\cline{1-1}\end{array}$
$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&7\\\hline 6\\\cline{1-1}\end{array}$
$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6&7\\\hline 5\\\cline{1-1}\end{array}$
$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline 7\\\cline{1-1}\end{array}$
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