March 30

Recap, finish formula from last week.

Formula:

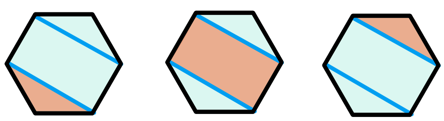
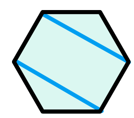
$T(n,k)$ = number of dissections of an n-gon by k cuts

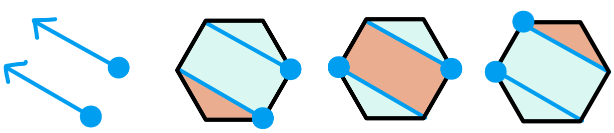
$$= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$$

1890 Cayley
... 2000 Przytycki, Sikora

Meaning of each part:

$\boxed{\frac{1}{k+1}}$ We overcount, then divide. k cuts $\Rightarrow$ $k+1$ regions.
Count k cuts with a marked region.

 counts  3 times

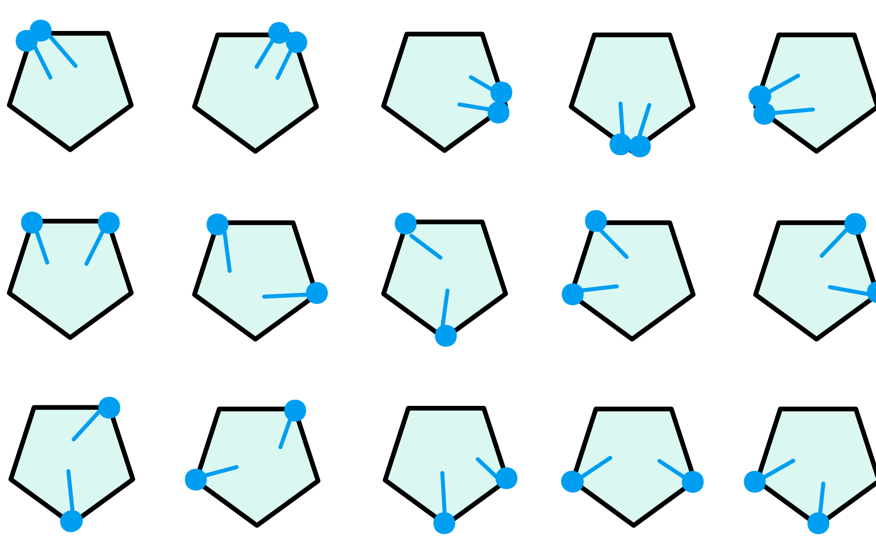


Orient each cut to keep marked region on the left.
Each cut now has a "start" •

$\boxed{\binom{n+k-1}{k}}$ # monomials of degree k in n variables

= # ways to start k cuts on n corners

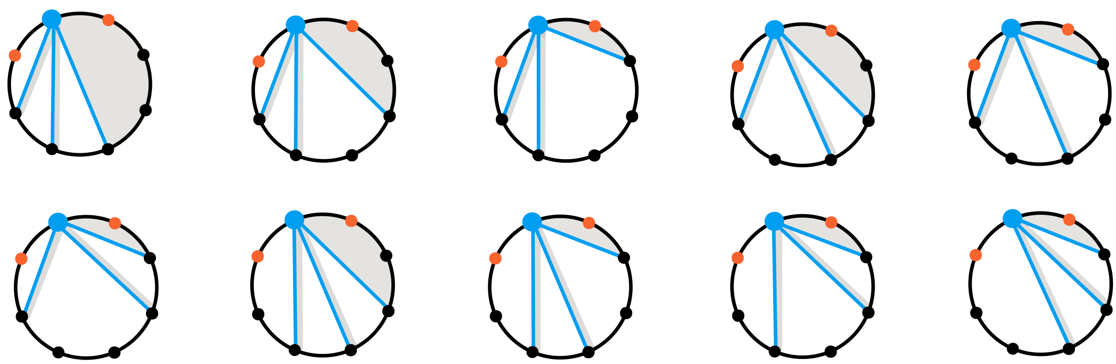
$n=5$, $k=2$ $\quad \binom{n+k-1}{k} = \binom{6}{2} = 15$



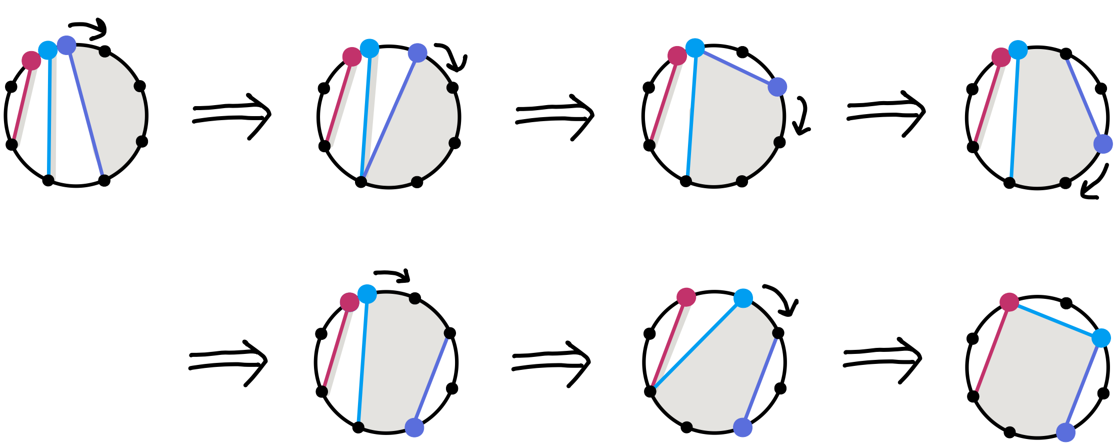
$\boxed{\binom{n-3}{k}}$ Counts ways to finish diagram, so cut directions are compatible with a choice of marked region.

Easy to see if all cuts start at same corner: There are n−3 eligible corners, we pick k of them.

$n=8, \quad k=3 \qquad \binom{n-3}{k} = \binom{5}{3} = 10$

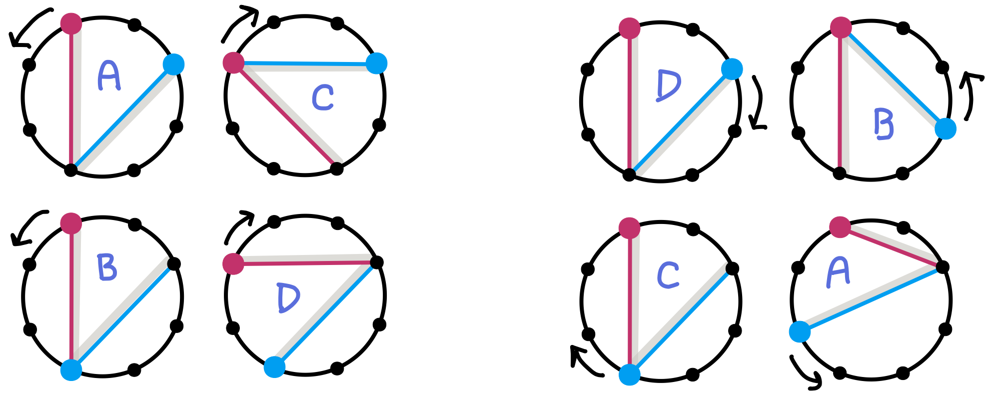


Modification of 2000 Przytycki, Sikora argument: Slide starting positions around like abacus beads. Transfer above configurations by reversible steps to any set of starting positions,



Usually we just rotate a cut to move its start. When two cuts collide, unique way to resolve conflict so there is still a consistent marked region.

Two kinds of transitions:

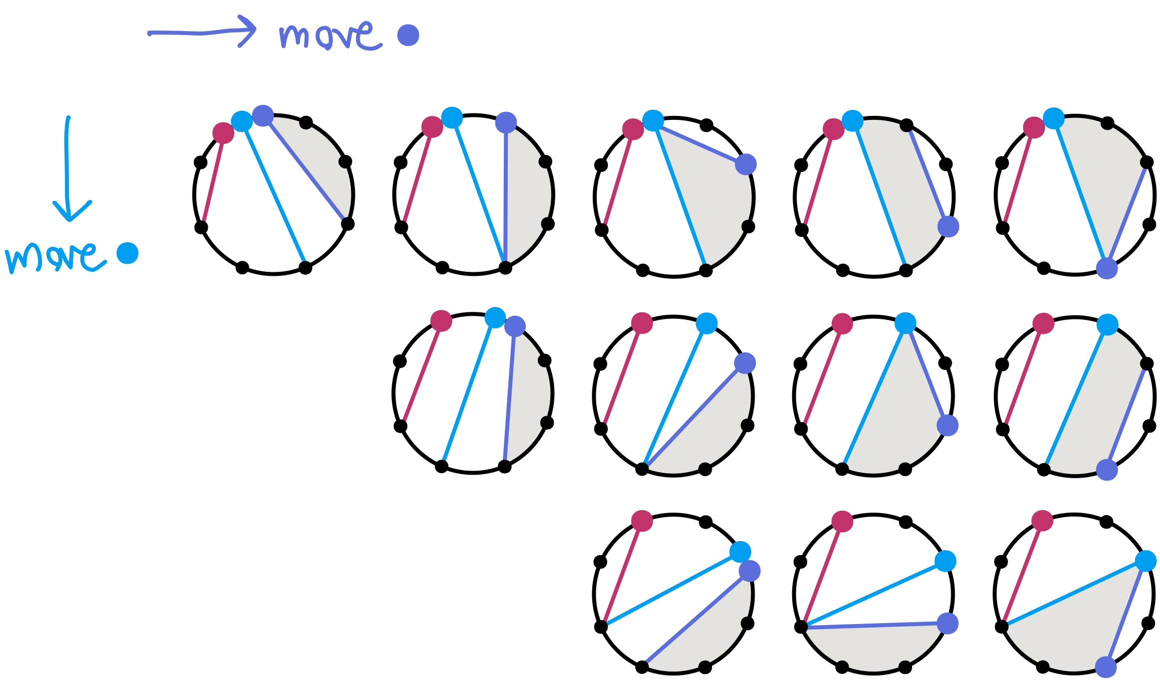


Other cases don't arise:

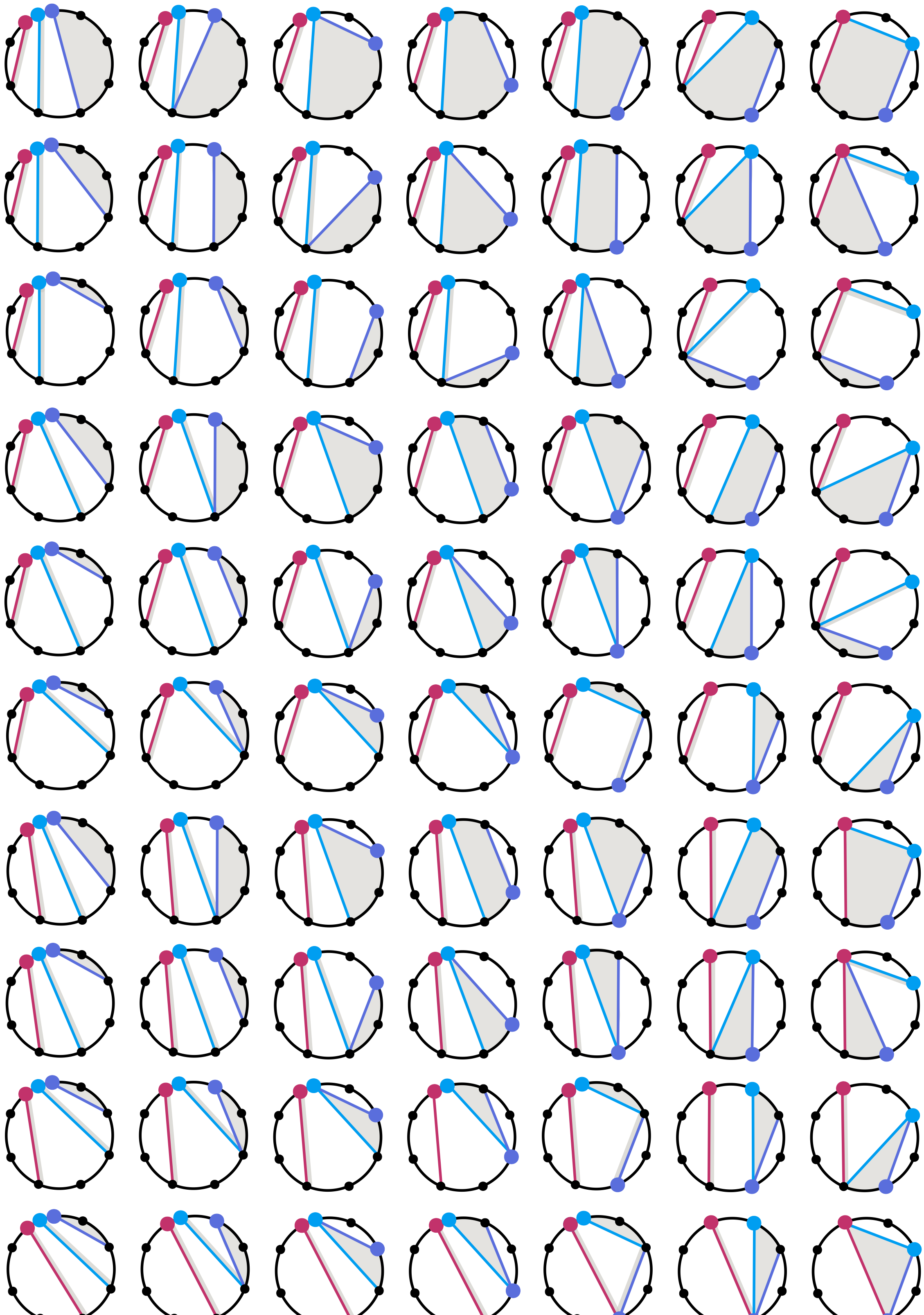
We don't let starts pass through each other

not consistent

The order in which we move starts doesn't matter.
Not that we need to care. We get a 1:1 correspondence by retracing our steps.



Example correspondence for n=8, k=3

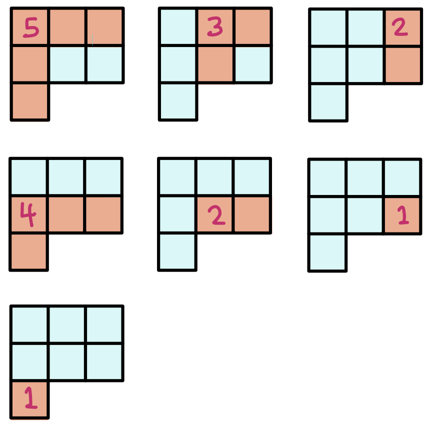


Young tableaux

Hook length formula

For each cell, record the length of the "hook" down or over.

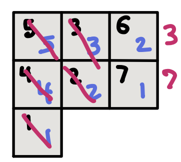
5	3	2
4	2	1
1		



For n cells, divide n! by the product of the hook lengths.

$$\frac{7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{5}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{4}\cdot\cancel{2}\cdot 1\cdot 1} = 21$$

or

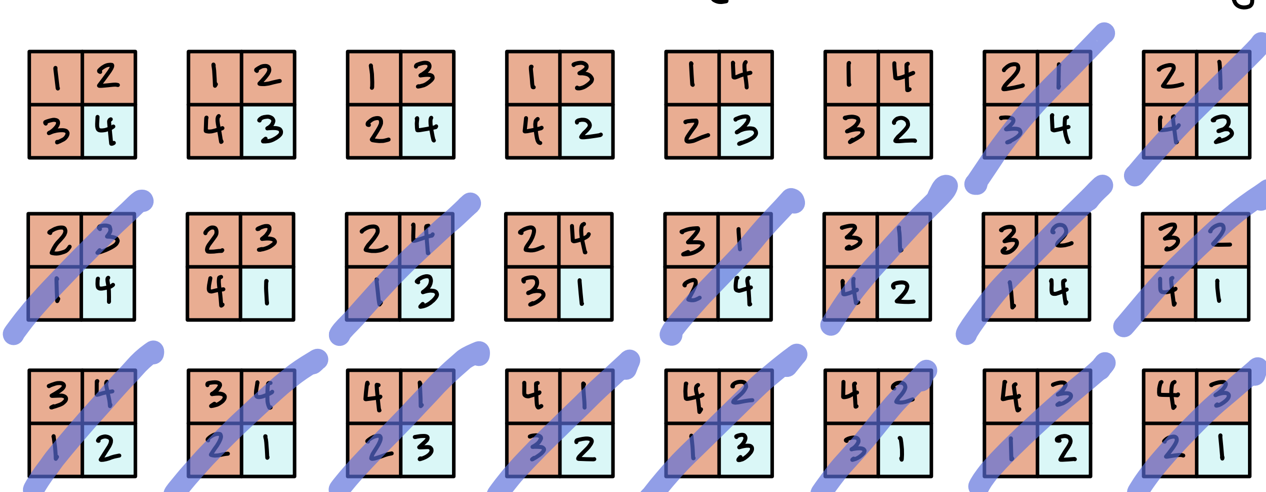


Still no proof that makes this obvious.

Knuth's heuristic argument:

Fill in tableau at random, and look at one hook.

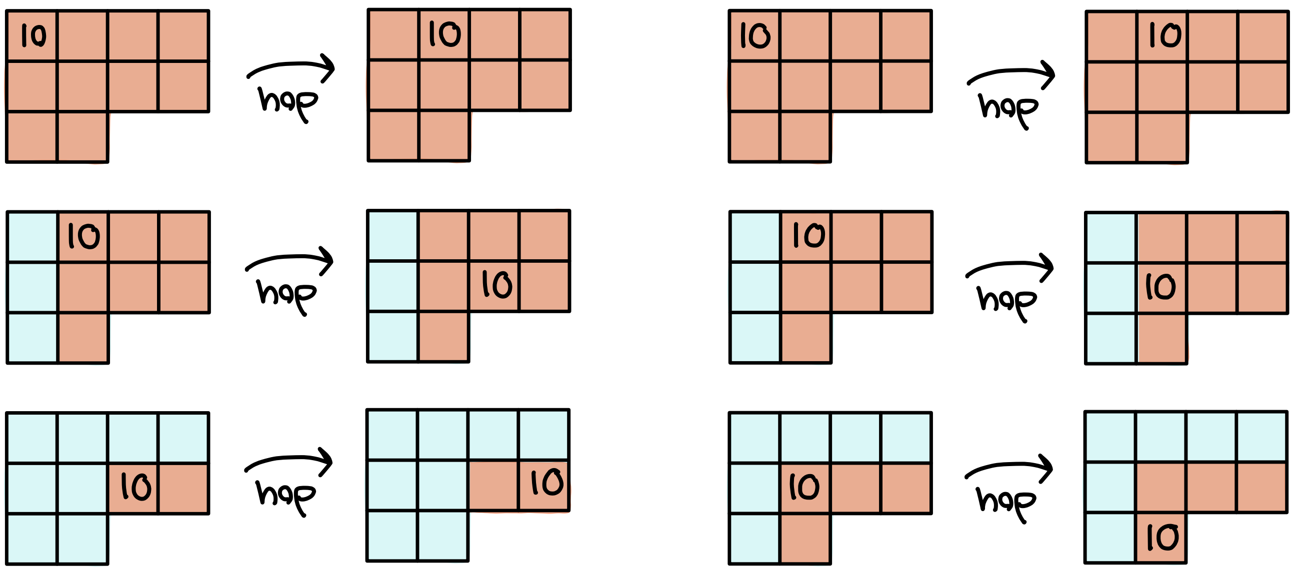
Chances of smallest element being at corner = 1/hook length



Problem: These probabilities aren't independent, for diffent hooks.

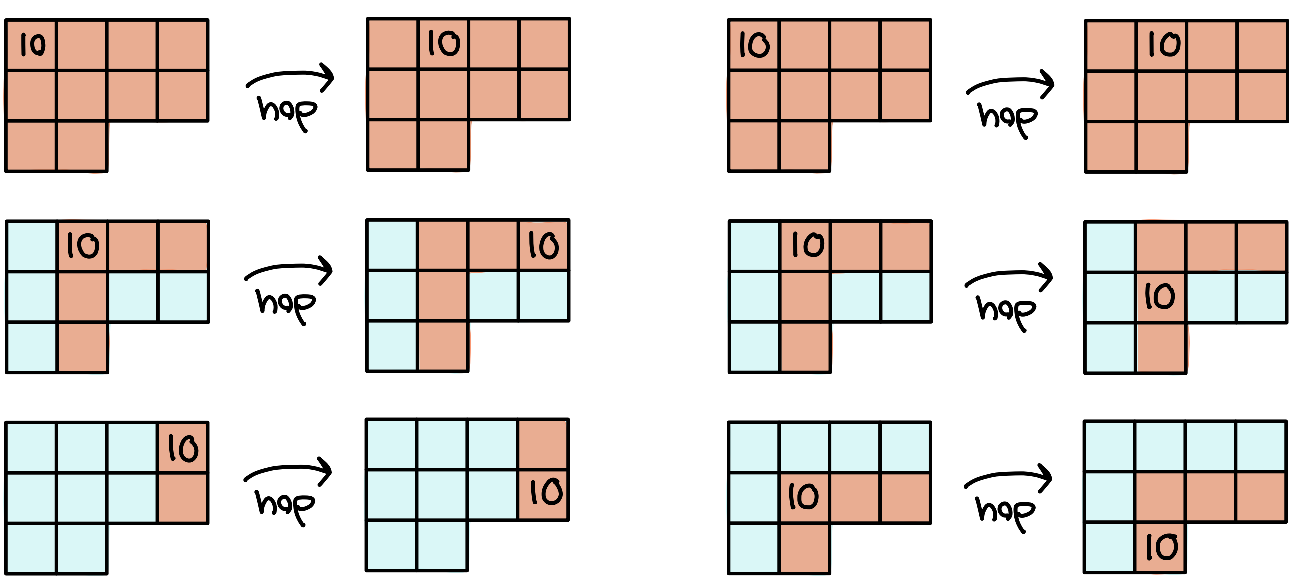
My idea in college (while taking a course with Herb Wilf):

Generate a Young tableau at random,
Start with n in upper left corner
Hop down/over uniformly at random till stuck.
Now iterate. Position n-1, then n-2, then...

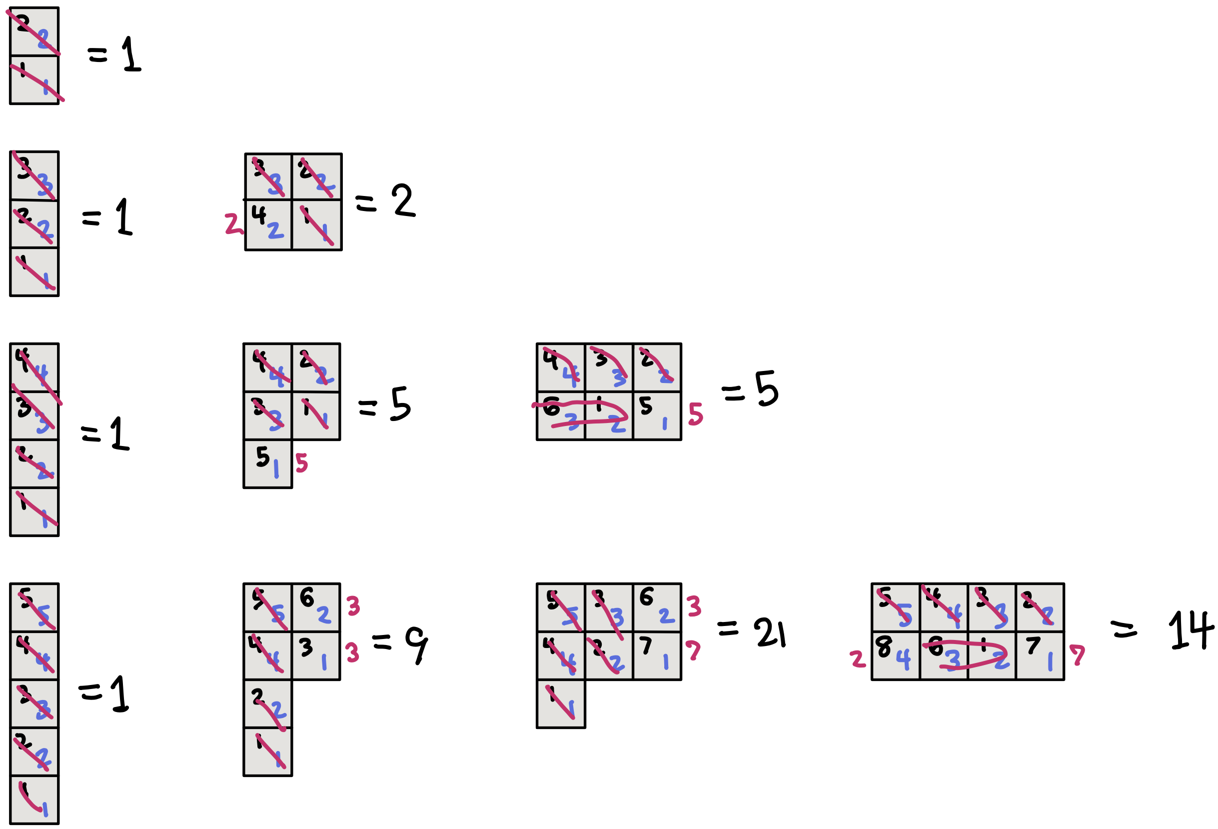


This doesn't quite work. Um, hook lengths? I still kick myself.

1979 Greene, Nijenhuis, Wilf came up with a better process:
After the first step, jump within hooks. Leads to proof of formula.



Special case: Two equal rows, one column

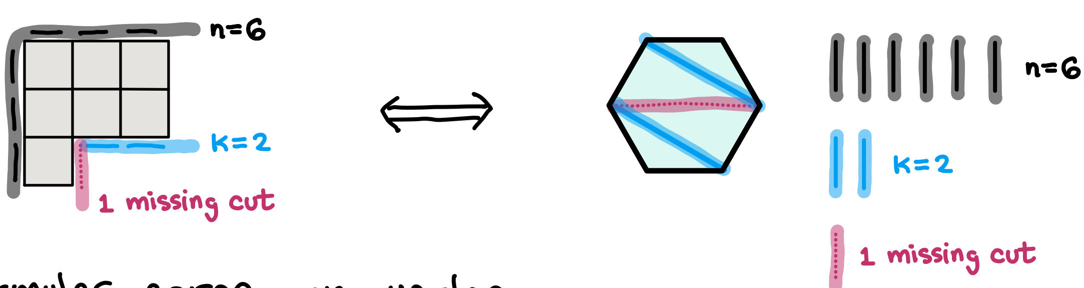


Compare:

$T(n, k)$ = number of dissections of an n-gon by k cuts

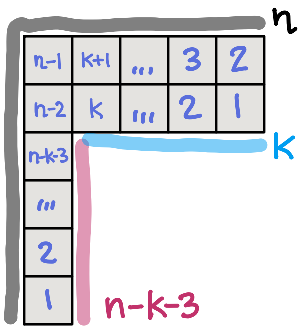
n-gon \ k cuts	0	1	2	3	4	5	6
3	1						
4	1	2					
5	1	5	5				
6	1	9	21	14			
7	1	14	56	84	42		
8	1	20	120	300	330	132	
9	1	27	225	825	1485	1287	429

Catalan numbers



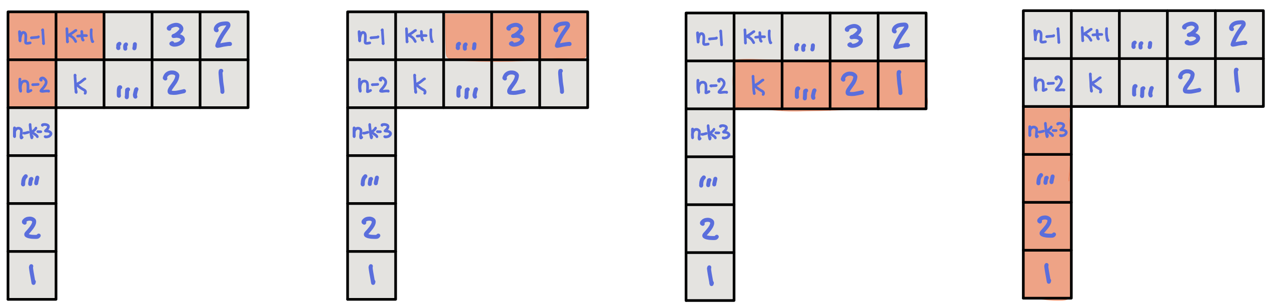
Formulas agree, on overlap.
These sets are in 1:1 correspondence.

Formulas agree, on overlap:



hook lengths

$\underbrace{2k + (n-k-3) + 2}_{n+k-1}$ cells



$(n-1)(n-2)(k+1)$ $k!$ $k!$ $(n-k+3)!$

$$\frac{(n+k-1)\cdots n\,(n-1)(n-2)\,(n-3)\cdots(n-k+4)\,(n-k+3)\cdots 3\cdot 2\cdot 1}{(k+1)\;k!\;(n-1)(n-2)\;k!\;(n-k+3)!}$$

$$= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$$

- Good that formulas agree
- Better to find 1:1 correspondence between sets
- Even better if correspondence:
  - has low complexity
  - preserves a neighbor graph ...
  - preserves a polytope

Here, we could learn more about Young tableaux from what we know about polygon dissections.