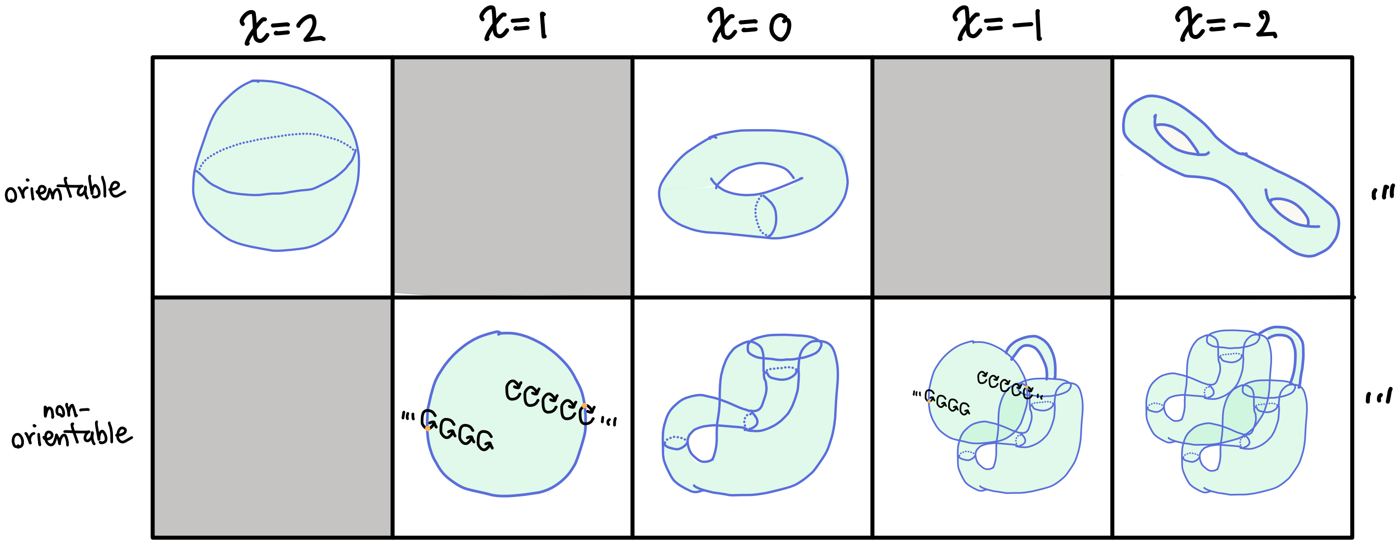
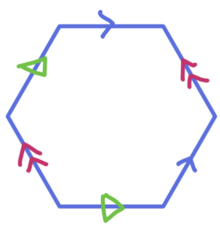
April 13 Graph Colorings

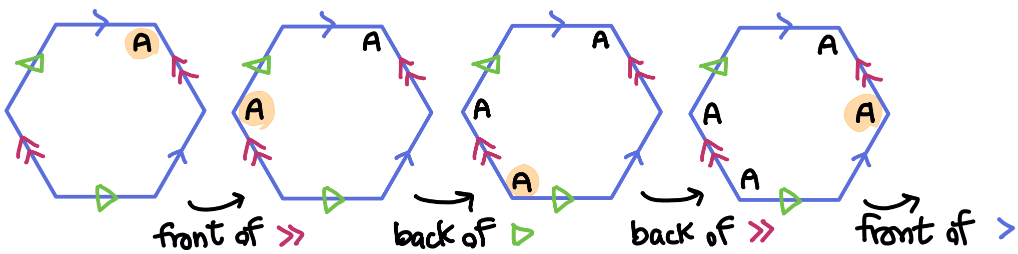


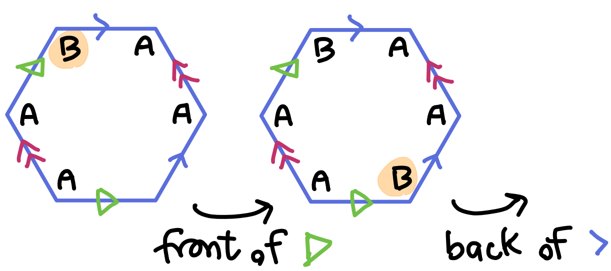
Identifying surfaces from their gluing diagrams



$$\chi = \underline{v - e + f}$$
$\quad$? $\quad$ 3 $\quad$ 1

chase identifications to enumerate vertices



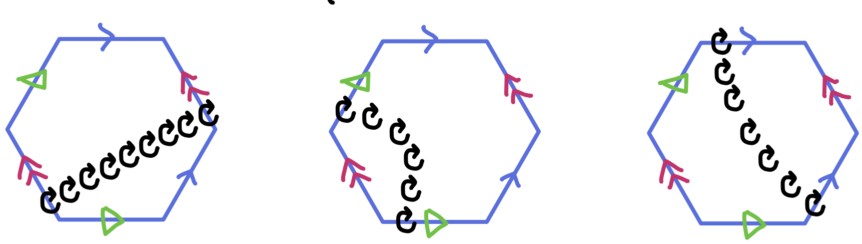


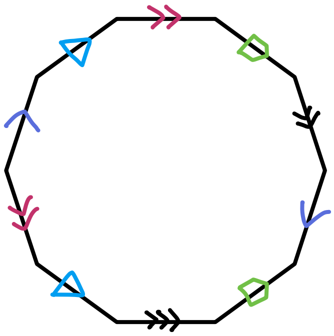
... so $V=2$ $\quad$ A,B

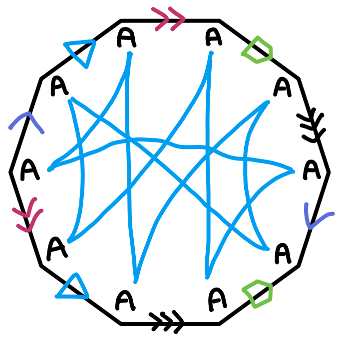
$\chi = 2-3+1 = 0$

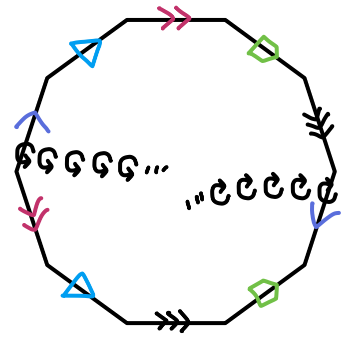
torus or Klein bottle

Is it orientable? No loops that reverse orientation. $\Rightarrow$ torus





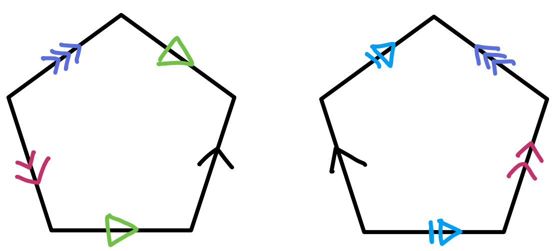


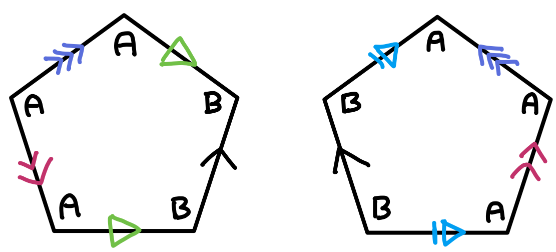


$\chi = 1 - 5 + \underbrace{1 = -3}_{\text{This alone tells us can't be orientable}}$

orientation-reversing loop

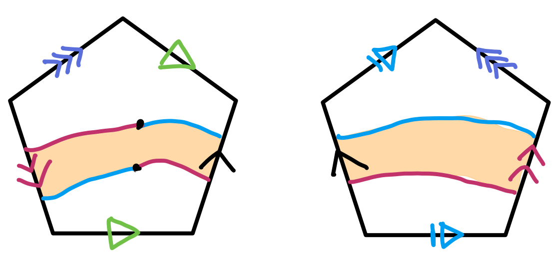
We can also glue multiple pieces:



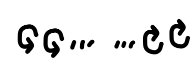


$\chi = 2 - 5 + 2 = -1$

Again must be non-orientable



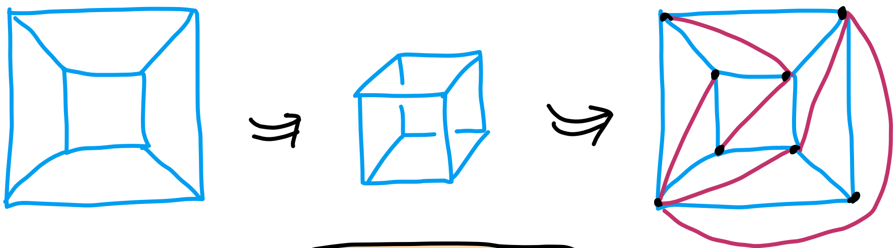
We can find Möbius strip inside surface

Same idea as 

Apply Euler characteristic $\chi = v - e + f$ to planar graphs:

$$\boxed{\text{Every planar graph has a vertex of degree} \leq 5}$$

① View a planar graph as drawn on a sphere ($\chi = 2$)
Make extra cuts so graph is triangulation of the sphere



	v	e	f
$\chi =$	8	-18	$+12 = 2$

$$\boxed{v - e + f = 2}$$

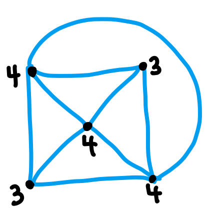
② Every triangle has 3 sides. This counts every edge twice.

$$\boxed{2e = 3f}$$

③ Let d be the average degree of a vertex.
Each edge has two ends, so dv counts each edge twice.

$$\boxed{dv = 2e} \Rightarrow e = dv/2, \; f = dv/3$$

Check these equations in an example:



v	e	f	d
5	9	6	18/5

$$\tfrac{1}{5}(4+4+4+3+3) = \tfrac{18}{5}$$

$$\underset{5}{v} - \underset{9}{e} + \underset{6}{f} = 2$$

$$\underset{2\cdot 9}{2e} = \underset{3\cdot 6}{3f}$$

$$\underset{\frac{18}{5}\cdot 5}{dv} = \underset{2\cdot 9}{2e}$$

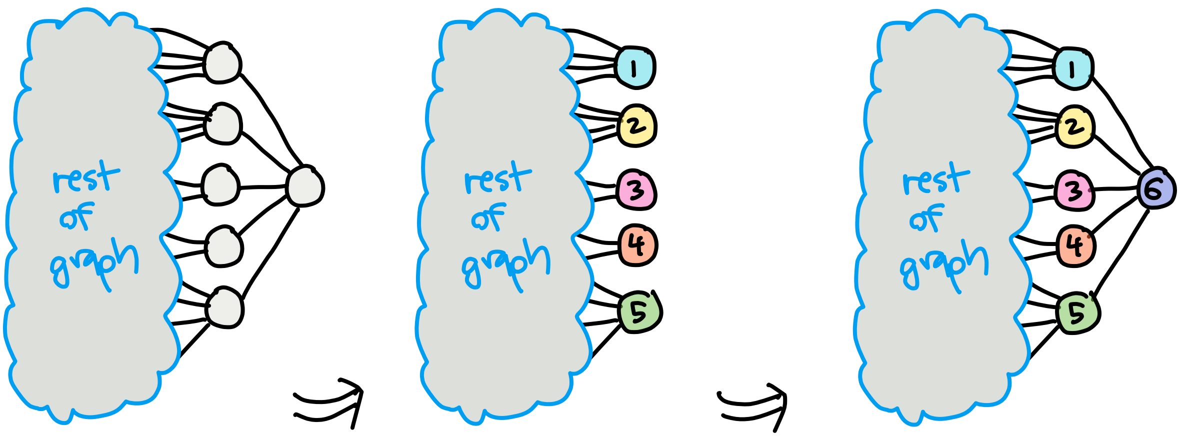
$$2 = v - e + f = v - dv/2 + dv/3$$
$$= v(\underbrace{1 - d/6}_{\text{must be positive}}) \Rightarrow \boxed{d < 6}$$

Every planar graph can be colored using 6 colors

Inductive proof / recursive algorithm:

Find a vertex of degree ≤ 5
Delete it, and 6-color the smaller graph left
Now add it back, and choose a color not used by its neighbors

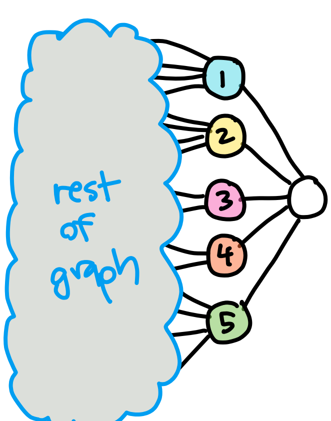
$\{ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \}$



... Every planar graph can be colored using 5 colors

$\{ 1 \ 2 \ 3 \ 4 \ 5 \}$

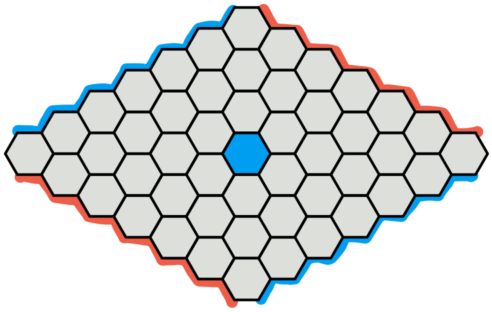
Same proof, but we need an idea to handle this case:
(we're out of colors)

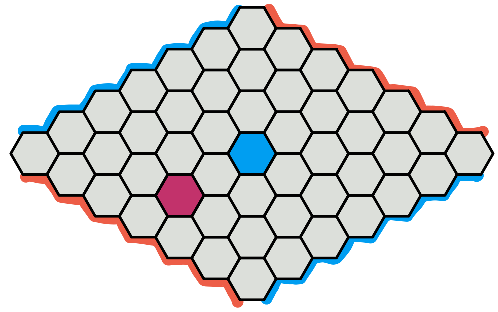


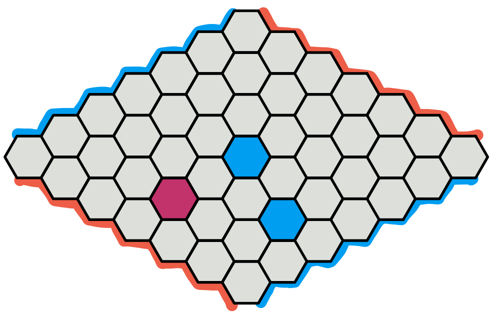
what do we do?

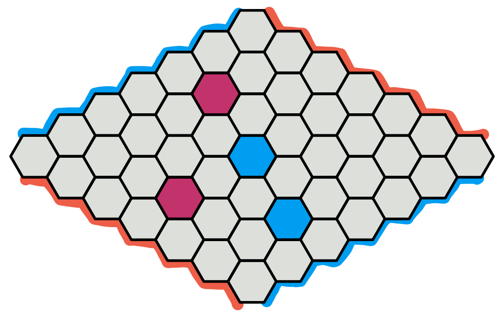
Surface topology enters again

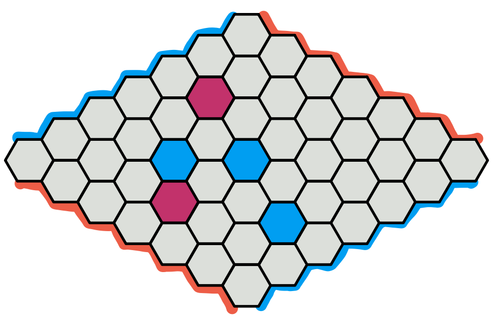
Same idea as proof some player wins Nash / Hex

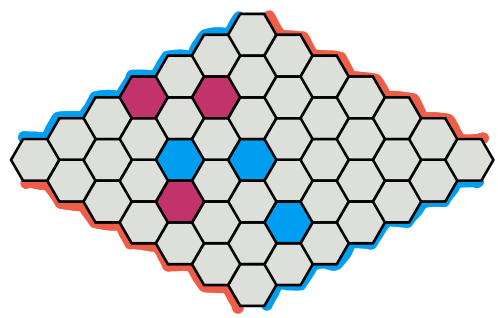


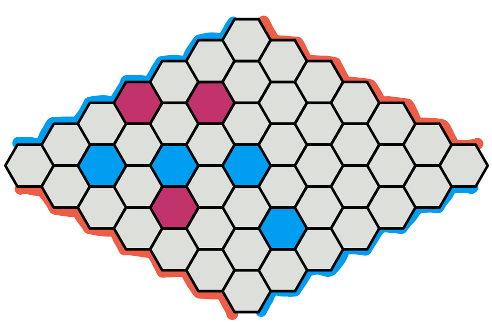




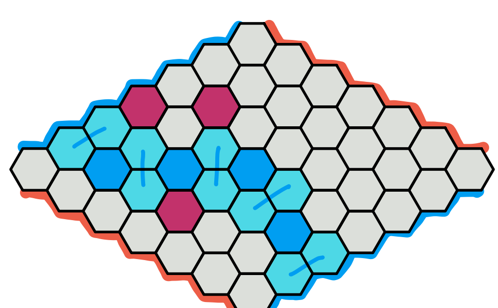








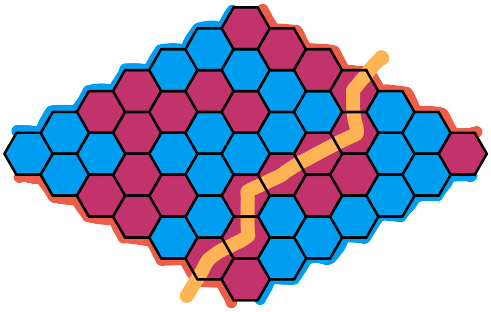
... Red resigns



Blue has a forced win

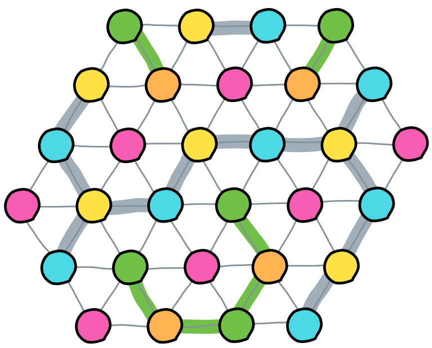
(14x14 is much harder)

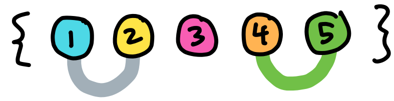
Topology: If we color every cell, Red or Blue has a win



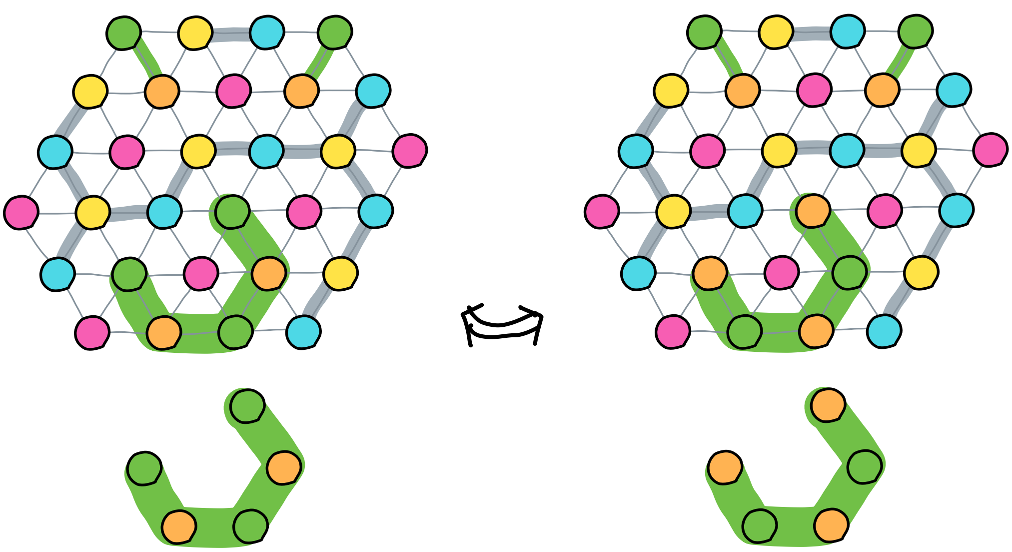
A bridge Red to Red blocks any possible bridge Blue to Blue.

Same with chains of colors in a planar graph.

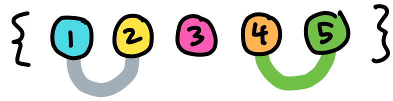


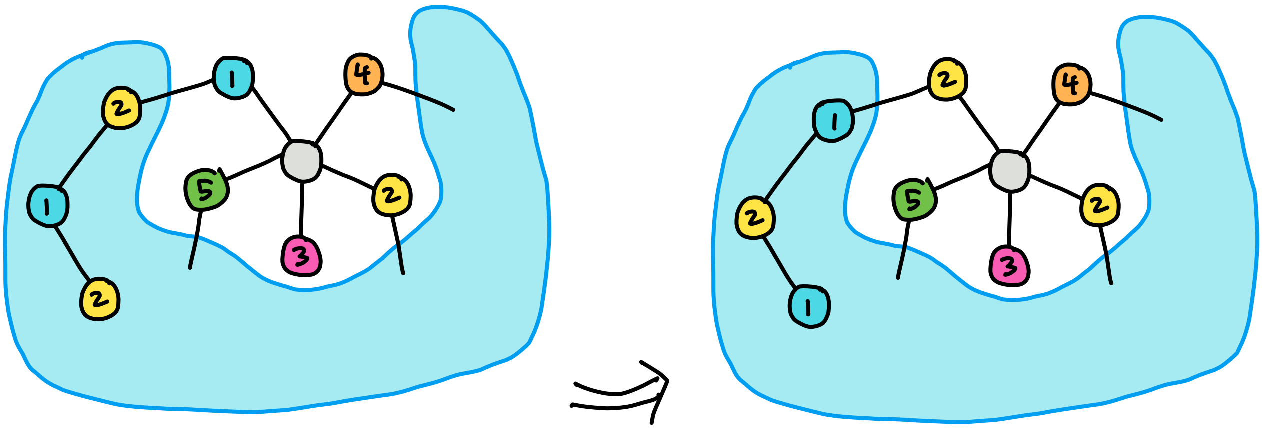


Chains block each other

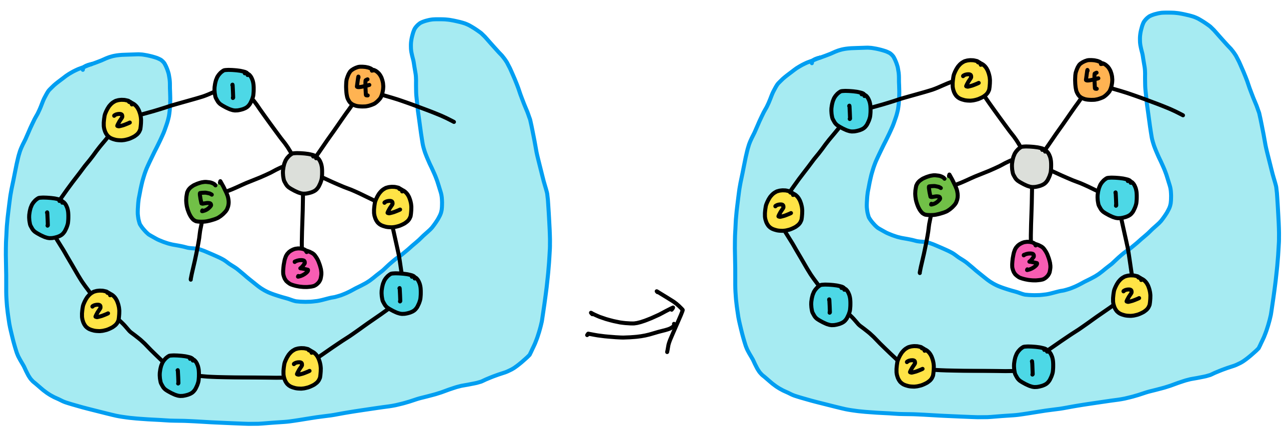


We can swap the colors in a connected chain, and no one else cares

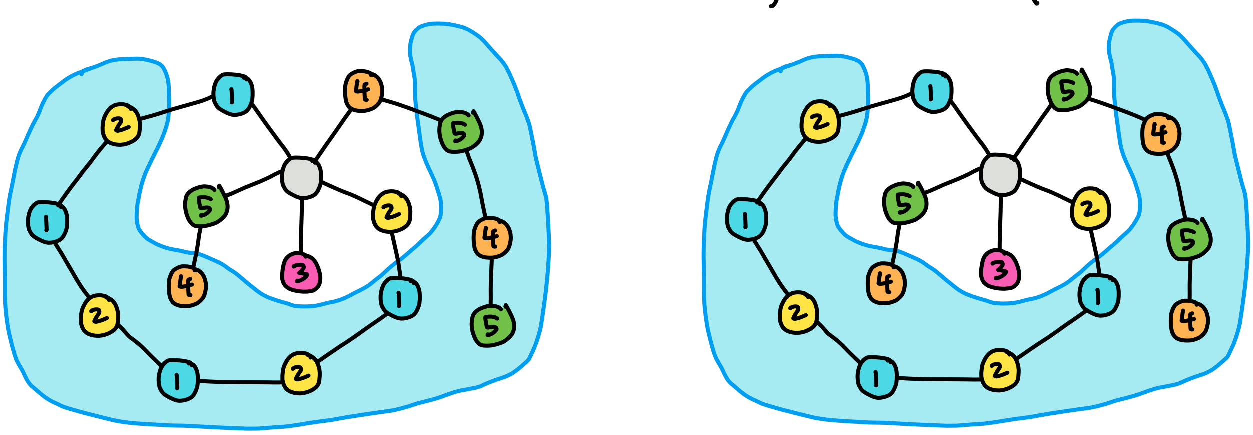




Swap 1 2 to make room



Doesn't work if they form a loop



If so, loop separates 4 5 so we can toggle a 4 5 strand.