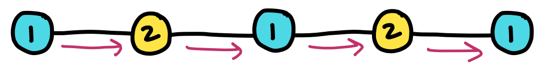
Chromatic polynomial

{ 1 2 3 4 5 }

Let $f(n)$ = # ways to color a graph G using up to n colors

$f(n)$ is a polynomial in n

G is a vine on K vertices:

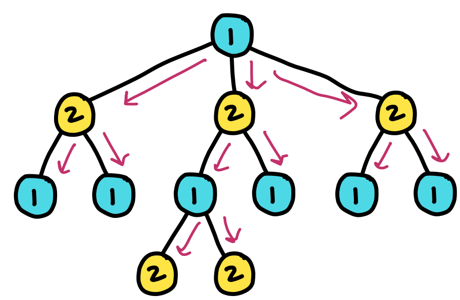


First vertex can be any color: n choices

Each following vertex can't repeat previous color: $(n-1)$ choices

$$f_K(n) = n(n-1)^{K-1}$$

G is a tree on K vertices:

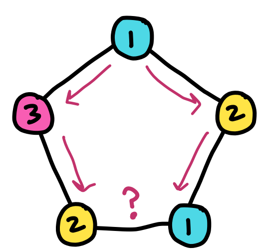


Same argument

Grow tree one vertex at a time

$$f_K(n) = n(n-1)^{K-1}$$

G is a cycle on K vertices:

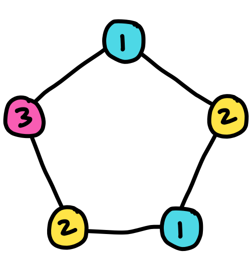


Ignore bottom edge

⇒ too many colorings

Subtract invalid colorings

= valid colorings if we collapse edge



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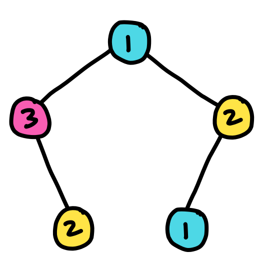

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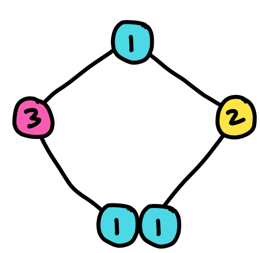
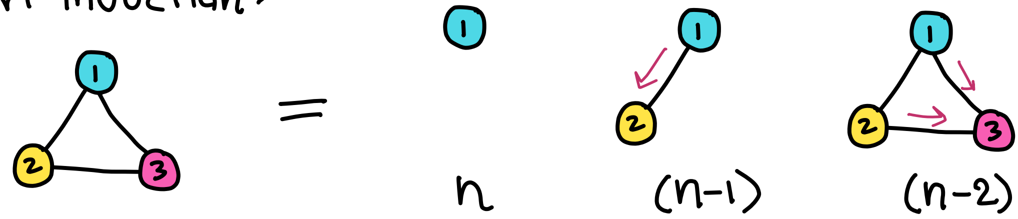


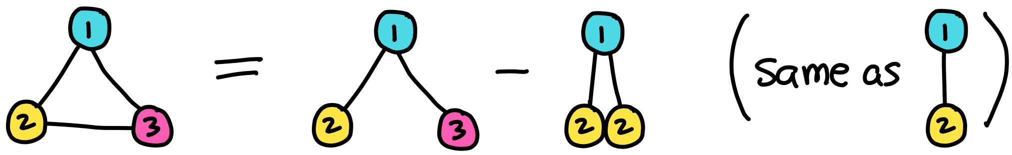
$$f_K(n) = n(n-1)^{K-1} - f_{K-1}(n)$$

Start induction:

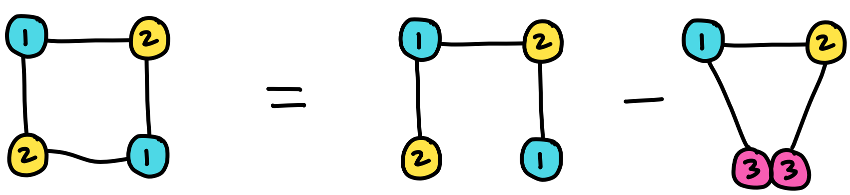


$$f_3(n) = n(n-1)(n-2)$$

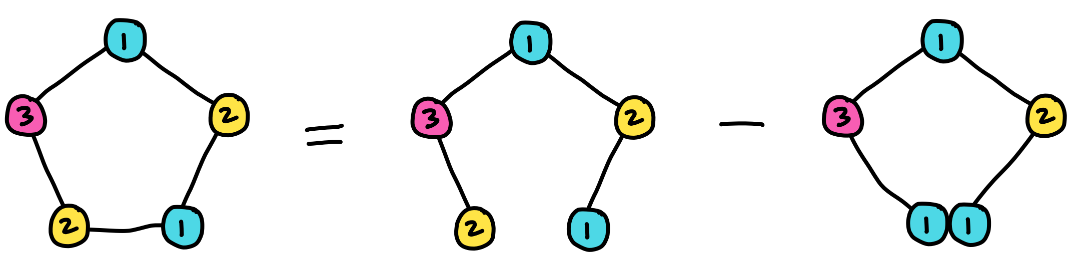
Or recurse, see what happens



$$\begin{aligned} f_3(n) &= n(n-1)^2 - n(n-1) \\ &= n(n-1)[(n-1)-1] = n(n-1)(n-2) \end{aligned}$$



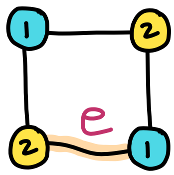
$$\begin{aligned} f_4(n) &= n(n-1)^3 - n(n-1)(n-2) \\ &= n(n-1)[(n-1)^2-(n-2)] \end{aligned}$$



$$\begin{aligned} f_5(n) &= n(n-1)^4 - n(n-1)[(n-1)^2-(n-2)] \\ &= n(n-1)[(n-1)^3-(n-1)^2+\underbrace{(n-1)-1}_{\text{better way to show pattern}}] \end{aligned}$$

General construction: Deletion / Contraction

Graph G, edge e



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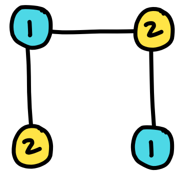
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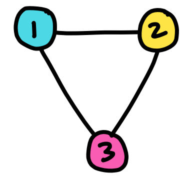
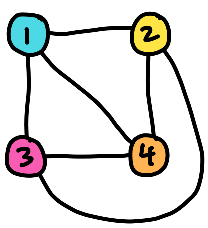


G $\qquad$ $G\setminus e$ $\qquad$ G/e

$$f_G(n) = f_{G\setminus e}(n) - f_{G/e}(n)$$

Example: Complete graph on 4 vertices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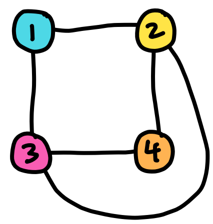
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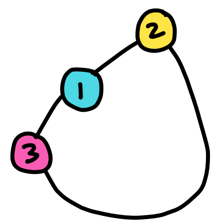

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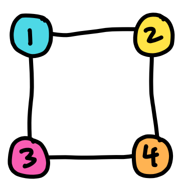
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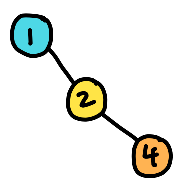
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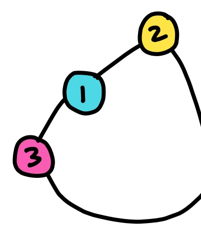
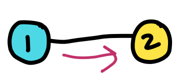
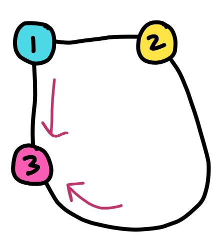
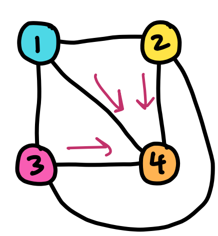


$$n(n-1)\left[(n-1)^2-(n-2)\right] - n(n-1)^2 - n(n-1)(n-2)$$

$$= n(n-1)\big[\underbrace{(n-1)^2-(n-2)-(n-1)-(n-2)}_{n^2-5n+6\ =\ (n-2)(n-3)}\big]$$

$= n(n-1)(n-2)(n-3)$ as expected

n $\qquad$ $(n-1)$ $\qquad$ $(n-2)$ $\qquad$ $(n-3)$

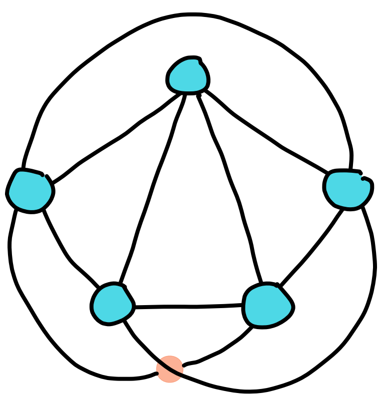
Graph Minors

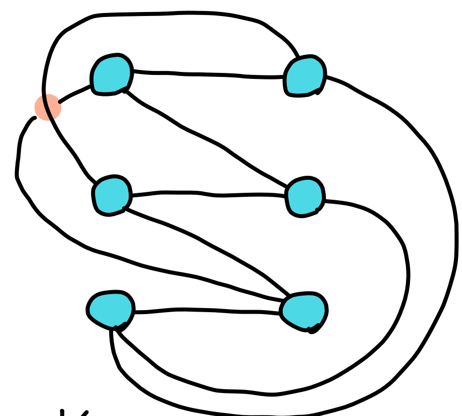
A graph H is a minor of a graph G
$\Leftrightarrow$ H can be obtained from G by deletion and contraction.

History:

Kuratowski's theorem (1930): A graph G is planar $\Leftrightarrow$ it does not contain a subdivision of K_5 or $K_{3,3}$

Wagner's theorem (1937): A graph G is planar $\Leftrightarrow$ it does not contain K_5 or $K_{3,3}$ as a minor



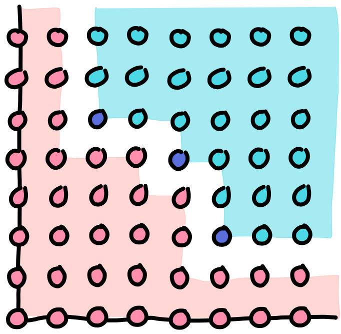


Interpretation: A property of graphs is minor closed if it is preserved by taking minors.
A graph not having this property is minor minimal if each of its minors has the property.

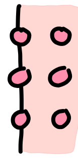
- Any minor of a planar graph is also planar.
- K_5 and $K_{3,3}$ are the minor minimal nonplanar graphs

So Kuratowski / Wagner theorem is a finiteness theorem.

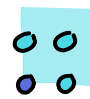
Hilbert basis theorem (1890)



Think of ↰ as deletion/contraction



is minor closed



↰ is minor minimal

Graph minors are same idea:

Robertson–Seymour theorem (1983–2004, 500 pages)

Every minor closed property of graphs can be characterized by a finite set of forbidden minors.

⇕

Everyone has heard of Everest but K2 is a bigger deal.

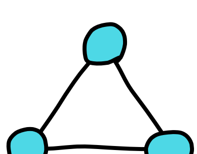
⇕

Four color theorem

1976 Appel, Haken
1997 Robertson, Sanders, Seymour, Thomas
2005 Gonthier

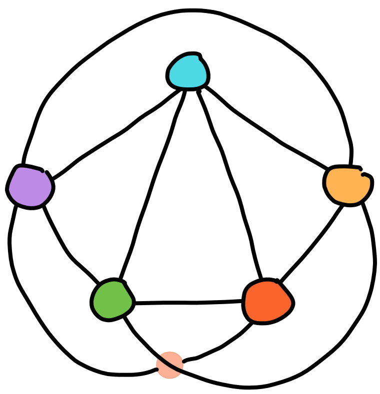
Other examples:

- If G can be drawn on a $\underbrace{\text{surface}}_{\text{plane, sphere, torus,...}}$, then so can any minor
  forbidden minors complicated / unknown
- If G is a forest (union of trees ⇔ no cycles) then so is any minor.



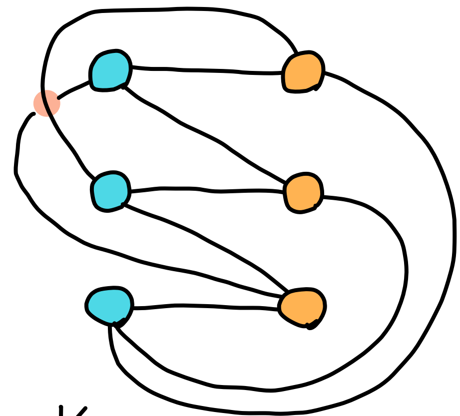
The triangle is unique forbidden minor

Four color theorem



K_5

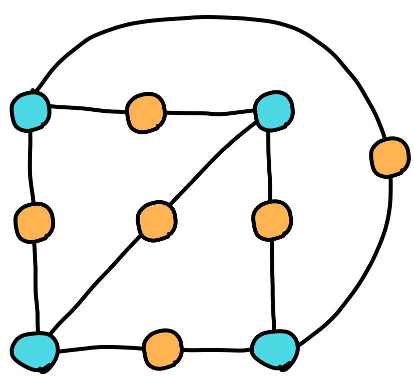
bad



$K_{3,3}$

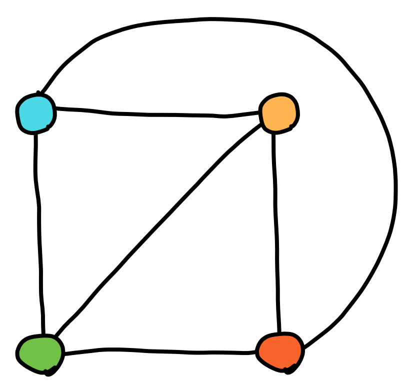
ok

Being n-colorable is not minor closed:



2-colorable

$\Rightarrow$



4-colorable

Hadwiger's conjecture $\Rightarrow$ **Four color theorem**

If G is loopless and contains no K_{n+1} minor, then G is n-colorable.

Known for $1 \leq n \leq 5$

Matroid theory

Let $v_1, \ldots, v_K$ be vectors in the vector space W over the field K

Let $\mathcal{S}$ = the set of subsets $B \subset \{v_1, \ldots, v_K\}$ that are linearly independent

How can we spot a valid $\mathcal{S}$?

Easily seen rules not quite enough. Exchange axiom:

If A, B independent sets and $|A| > |B|$ then $\exists v \in A \setminus B$ so $B \cup \{v\}$ is independent

An abstract matroid (data for $\mathcal{S}$ satisfying these rules) is representable over the field K

$\Leftrightarrow$ we can find $v_1, \ldots, v_K$ vectors in the vector space W over K, with this structure $\mathcal{S}$

- Can a graph G be drawn on a surface S?
  Forbidden minors, depending on S

- Can a matroid $\mathcal{S}$ be represented over a field K?
  Forbidden minors, depending on K.

Definitions are more technical, but theory is entirely parallel.

Graphs are actually a special case of matroids:
A set of edges is independent $\Leftrightarrow$ they don't contain a cycle.

There is a chromatic polynomial for matroids.