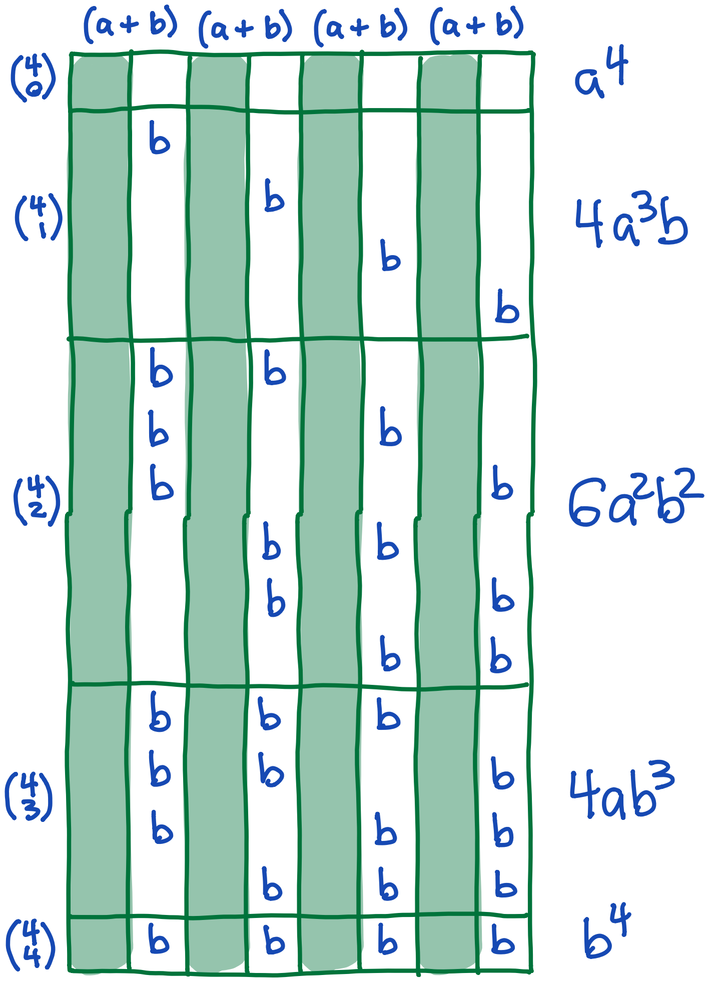
Generating functions

Prototype: Binomial theorem



$$(a+b)^4 =$$
$$\binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^3$$

4 terms in product
2 choose b
rest choose a

General pattern:

Algebra does a combinatorial dance we want to harness.

Monomials in two variables

$(1+x+x^2+x^3+x^4+\dots)\cdot$
$(1+y+y^2+y^3+y^4+\dots)$

$= 1$
$+ (x+y)$
$+ (x^2+xy+y^2)$
$+ (x^3+x^2y+xy^2+y^3)$
$+ \dots$

convolution FFT

+					
1	y	y^2	y^3	y^4	...
x	xy	xy^2	xy^3	xy^4	
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4	
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4	
x^4	x^4y	x^4y^2	x^4y^3	x^4y^4	
$\vdots$					

Geometric series

$$1+x+x^2+x^3+x^4+\dots = \frac{1}{1-x}$$
$$1+y+y^2+y^3+y^4+\dots = \frac{1}{1-y}$$

So product is

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)$$

Recall proof:

$$(1+x+x^2+x^3+x^4+\dots)(1-x)$$
$$\begin{aligned} = \; & 1+x+x^2+x^3+x^4+\dots \\ & \;\; -x-x^2-x^3-x^4 \quad \dots \\ \hline & 1 \end{aligned}$$

setting $x=y=t$, product is

$$\left(\frac{1}{1-t}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}$$

$$\boxed{\frac{1}{(1-t)^2}} = 1+2t+3t^2+4t^3+\dots = \boxed{\sum_{n=0}^{\infty} f(n)t^n}$$

n	0	1	2	3	4	...
$f(n)$	1	2	3	4	5	...

These are same thing!

These are same thing!

$f(n) = $ #monomials of degree n in x,y

Monomials in three variables

$g(n) = $ #monomials of degree n in x,y,z

$f(n) = 1 = \binom{n}{0}$ $\quad x$
$f(n) = n+1 = \binom{n+1}{1}$ $\quad x,y$
$f(n) = \dots = \binom{n+2}{2}$ $\quad x,y,z$

$$(1+x+x^2+x^3+\dots)(1+y+y^2+y^3+\dots)(1+z+z^2+z^3+\dots)$$
$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)\left(\frac{1}{1-z}\right)\Bigg|_{x=y=z=t} = \boxed{\frac{1}{(1-t)^3}} = \boxed{\sum_{n=0}^{\infty} g(n)t^n}$$

n	0	1	2	3	4	...
$g(n)$	1	3	6	10	15	...

check: $1/(1-t)^2$

$\frac{1}{1-t}$

$\times$	1	2	3	4	...
1	1	2	3	4	
1	1	2	3	4	
1	1	2	3	4	
1	1	2	3	4	
$\vdots$					

$g(n) = \binom{n+2}{2}$

n balls
2 dividers

x^3 $\quad$ ●●●|| (x y z)

x^2z $\quad$ ●●||● (x y z)

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1   ← row n
  1   5  10  10   5   1
```

choose 2

$$\boxed{\text{we prefer } \frac{1}{(1-t)^3} \text{ to } \binom{n+2}{2}}$$

Generating function:

For any function $f: \mathbb{N} \to \mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$)

consider instead the series $\boxed{\sum_{n=0}^{\infty} f(n)\, t^n}$

Compare Laplace transform from ODE's

$$f(t) \implies F(s) = \boxed{\int_0^{\infty} f(t)\, e^{-st}\, dt}$$

$\sum_{n=0}^{\infty}$	$\int_0^{\infty} dt$	$f(n)$	$f(t)$	t^n	$(e^{-s})^t$

sum / use function / take power

William Feller

An Introduction to Probability Theory and its Applications

Volumes 1,2

Straddles these worlds

Cult status book

Example: Making change for 20¢ using a: 1¢, b: 2¢, c: 5¢

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1¢	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2¢	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11
5¢	1	1	2	2	3	4	5	6	7	8	10	11	13	14	16	18	20	22	24	26	29
						1	1	2	2	3	4	5	6	7	8	10	11	13	14	16	

1111112
111122
11222
2222
1111111111

	0	2	4	6	8	10	12	14	16	18	20
0	0	2	4	6	8	10	12	14	16	18	20
5	5	7	9	11	13	15	17	19			
10	10	12	14	16	18	20					
15	15	17	19								
20	20										

$$\left.\begin{matrix} 11 \\ 8 \\ 6 \\ 3 \\ 1 \end{matrix}\right\} \boxed{29}$$

all ways of getting within 20 using 2,5

finish with pennies

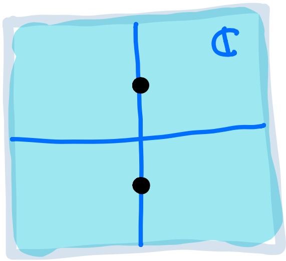
$$\left(\frac{1}{1-a}\right)\left(\frac{1}{1-b}\right)\left(\frac{1}{1-c}\right)\Bigg|_{\substack{a=t \\ b=t^2 \\ c=t^5}} = \boxed{\frac{1}{(1-t)(1-t^2)(1-t^5)}} = \ldots + \boxed{29t^{20}} + \ldots$$

Algebraic Geometry

Study geometry of zeros of polynomial systems of equations.

This is "bonus" material. It won't be on exams

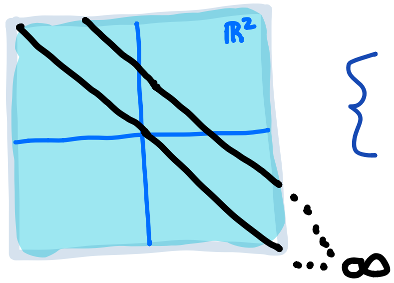
Need zeros!



$x^2+1=0$

Fix: work with $\mathbb{C}$ not $\mathbb{R}$

$(x+i)(x-i)=0$ zeros $i, -i$



$\left\{ \begin{matrix} x+y=0 \\ x+y=1 \end{matrix} \right\}$

Fix: work with projective space of ratios

$1:-1:0$

$\mathbb{R}^1 = \{x\}$ $\quad \mathbb{P}^1 = \{x:y\}$ ratio of x to y

$x \longmapsto x:1$

$\infty \longmapsto 1:0$

$\mathbb{R}^2 = \{(x,y)\}$ $\quad \mathbb{P}^2 = \{x:y:z\}$

$(x,y) \longmapsto x:y:1$ All possible ratios

$\mathbb{P}^1$ at $\infty \longmapsto x:y:0$ $x:y$ are points at ∞

$\left\{ \begin{matrix} x+y=0 \\ x+y=1 \end{matrix} \right\} \Longrightarrow \left\{ \begin{matrix} x+y=0 \\ x+y=z \end{matrix} \right\}$ "homogenize" using z

$1:-1:0$ is common solution at ∞

Ratios need homogeneous polynomials $\quad 1:-1:0 \approx 2:-2:0$ same ratio

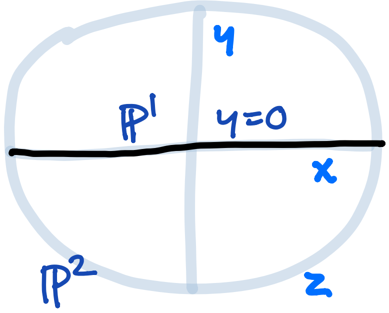
all terms same degree $d \iff f(\lambda x, \lambda y, \lambda z) = \lambda^d f(x,y,z)$ (both vanish or neither does)

Integers mod p : $m \approx n$ if they differ by a multiple of p

$\{0, 1, \ldots, p-1\}$

Polynomials mod a "variety" X (solution set) : $f \approx g$ if $f-g$ vanishes on X

Combinatorial examples



$\mathbb{P}^1$ ratios $x:y$
homogeneous polynomials in x,y

n	0	1	2	3	4	...
$f(n)$	1	2	3	4	5	...

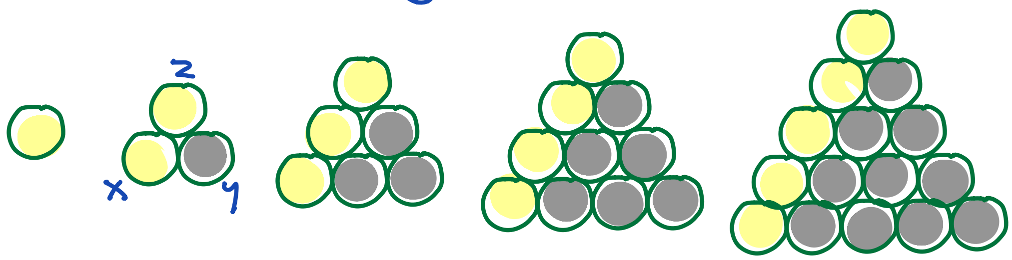
$$\boxed{\frac{1}{(1-t)^2}}$$

$\mathbb{P}^2$ ratios $x:y:z$
homogeneous polynomials in x,y,z

n	0	1	2	3	4	...
$g(n)$	1	3	6	10	15	...

$$\boxed{\frac{1}{(1-t)^3}}$$

What about $\mathbb{P}^1$ sitting inside $\mathbb{P}^2$ as $\{y=0\}$?



multiples of y, ≈ 0

n	0	1	2	3	4	...
$f(n)$	1	2	3	4	5	...

$$\boxed{\frac{1}{(1-t)^2}}$$

After modding by $X = \mathbb{P}^1$ defined by $y=0$, same answer.

Twisted cubic curve $\quad f: \mathbb{R} \hookrightarrow \mathbb{R}^3$
$$t \longmapsto (t, t^2, t^3)$$

use instead $\quad g: \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$
$$s:t \longmapsto s^3 : s^2t : st^2 : t^3$$

$s=1$

$a \quad b \quad c \quad d$ variables

Equations

$$\left\{\begin{aligned} b^2 &= ac \\ bc &= ad \\ c^2 &= bd \end{aligned}\right\} \qquad \begin{aligned} (s^2t)^2 &= (s^3)(st^2) \\ (s^2t)(st^2) &= (s^3)(t^3) \\ (st^2)^2 &= (s^2t)(t^3) \end{aligned}$$

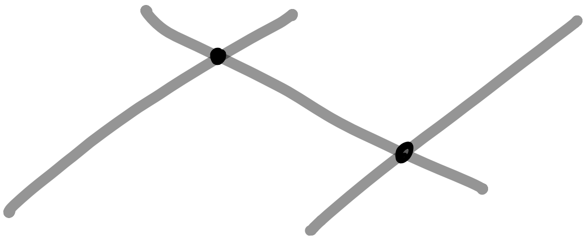
$21+21 = 30+12 = 42$ ✓
$21+12 = 30+03 = 33$ ✓
$12+12 = 21+03 = 24$ ✓

Monomials in a,b,c,d mod X (these equations)

$1 \;|\; a,b,c,d \;|\; a^2, ab, \underset{b^2}{ac}, \underset{bc}{ad}, \underset{c^2}{bd}, cd, d^2 \;|\; a^3, \ldots$

1   4   7   10

$$1 + 4t + 7t^2 + 10t^3 + \dots = \frac{3}{(1-t)^2} - \frac{2}{1-t}$$



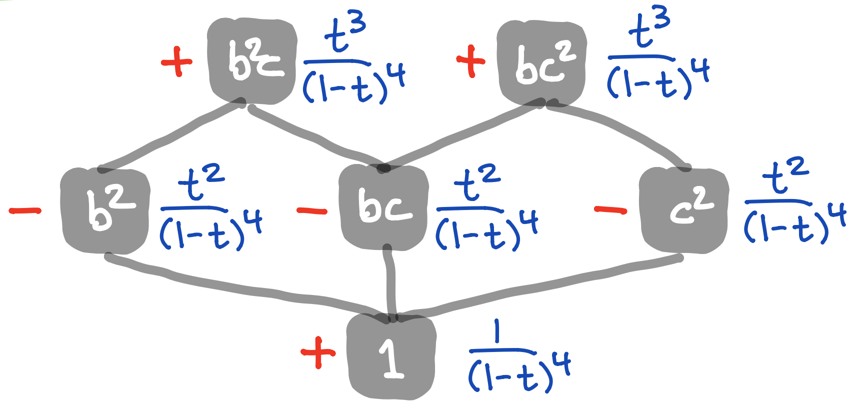
$$3\mathbb{P}^1 - 2\mathbb{P}^0$$

"size" of 3 lines − 2 points

A degenerate object like original curve

Leading terms give counting problem we've already studied:

How many monomials of degree n in a, b, c, d are not divisible by any of b^2, bc, c^2?



$$\frac{1 - 3t^2 + 2t^2}{(1-t)^4} = \frac{3}{(1-t)^2} - \frac{2}{1-t} = \boxed{3\mathbb{P}^1 - 2\mathbb{P}^0}$$

n	0	1	2	3	4	...
$\frac{1}{(1-t)^4}$	1	4	10	20	35	...
$\frac{t^2}{(1-t)^4}$			1	4	10	...
$\frac{t^3}{(1-t)^4}$				1	4	...

$$\boxed{\sum_{n=0}^{\infty} g(n) t^n}$$