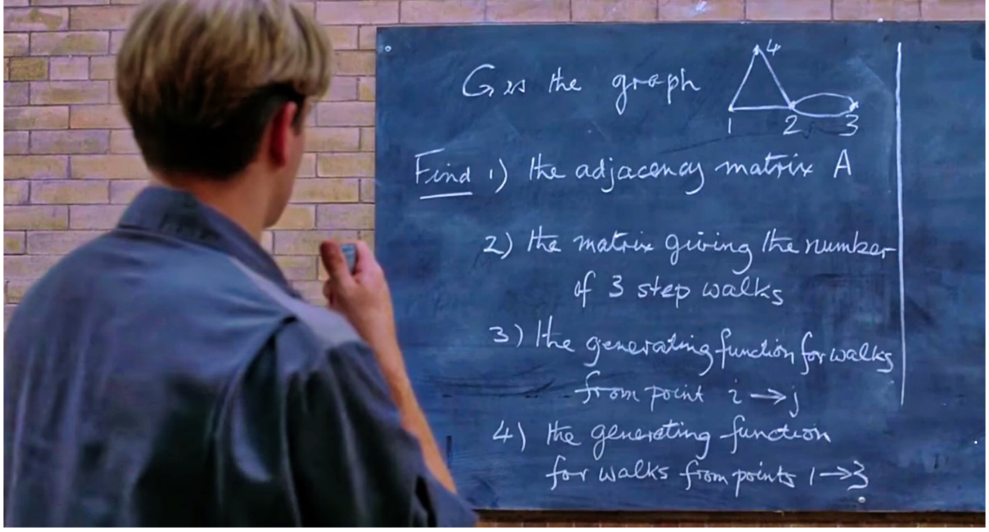

G is the graph

Find 1) the adjacency matrix A

2) the matrix giving the number of 3 step walks

3) The generating function for walks from point i→j

4) the generating function for walks from points 1→3

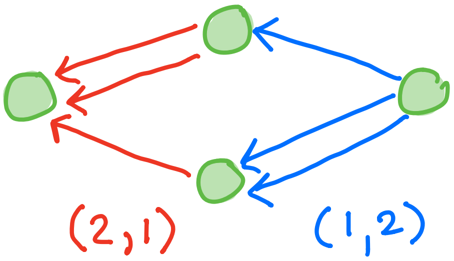
$\boxed{\text{Feb 2}}$

Good Will Hunting blackboard scene

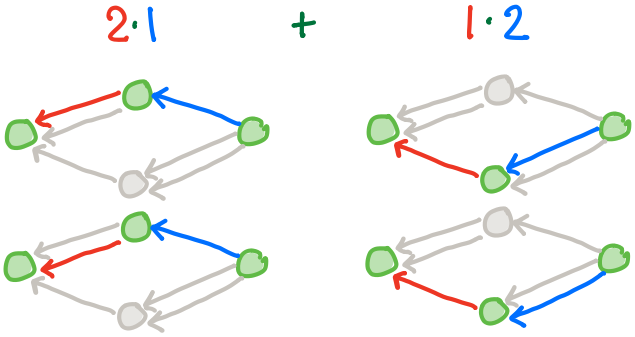
(Easy if you know what to do...)

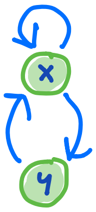
$\boxed{1}$ Matrix multiplication counts paths

$$(2,1)\cdot(1,2) = 2\cdot 1 + 1\cdot 2 = 4$$

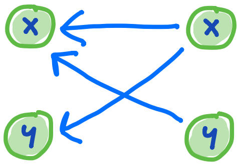


dot product



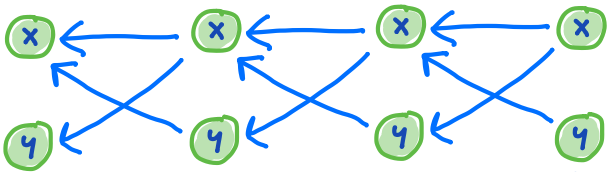


unfold as



$$\begin{array}{r|cc|l} & x & y & \text{start}\\ x & 1 & 1 & \\ \text{end }\ y & 1 & 0 & \end{array}$$

one step



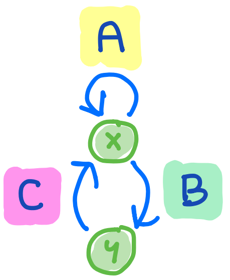
$$\begin{bmatrix}1&1\\1&0\end{bmatrix}\underbrace{\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}}_{\begin{bmatrix}2&1\\1&1\end{bmatrix}} = \begin{array}{r|cc|l} & x & y & \text{start}\\ x & 3 & 2 & \\ \text{end }\ y & 2 & 1 & \end{array}$$

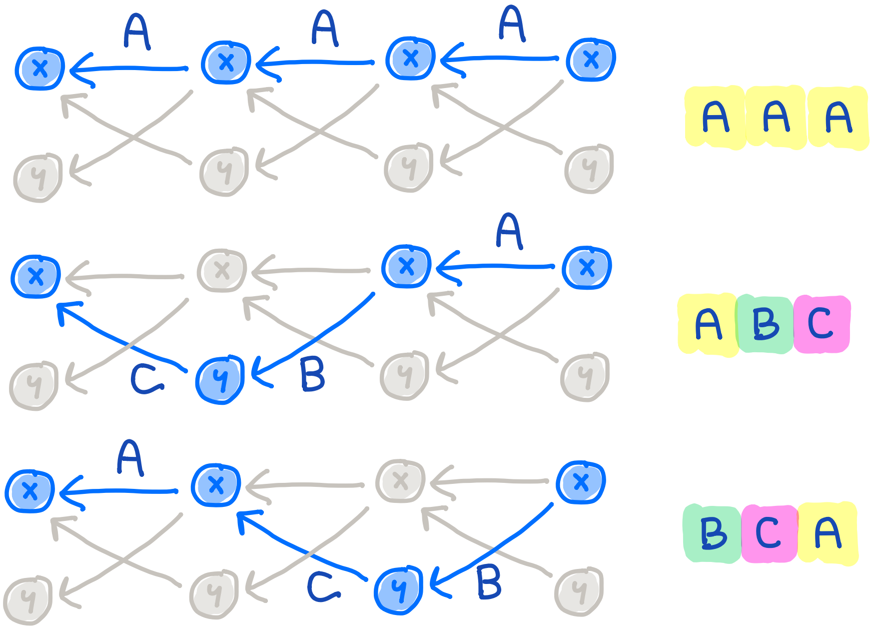
3 steps

Right to left is function composition order:

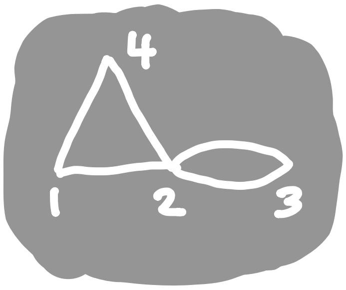
$f(g(x)) \qquad \underset{\Leftarrow\Leftarrow}{f\circ g}$

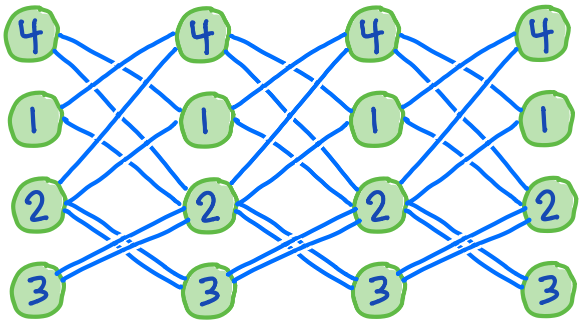
check:





$$\begin{array}{cc|cc|c} & & x & y & \text{start} \\ \hline & x & 3 & 2 & \\ \text{end} & y & 2 & 1 & \end{array}$$





1)

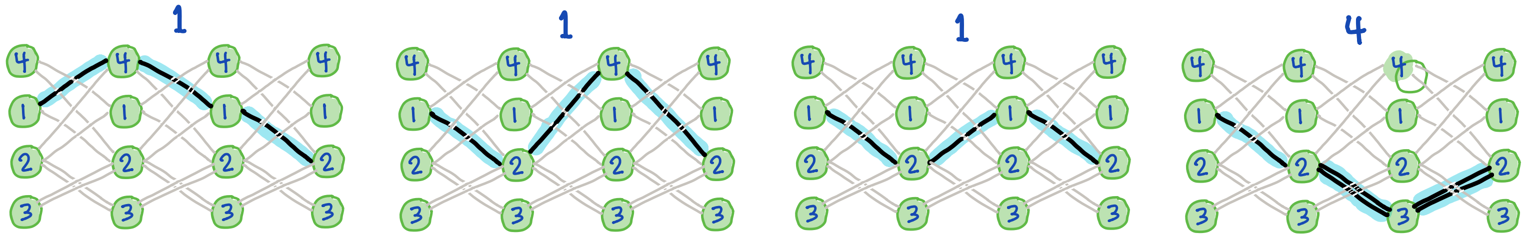
$$\begin{array}{cc|cccc|c} & & & & & \text{right} & \\ & & 1 & 2 & 3 & 4 & \\ \hline & 1 & 0 & 1 & 0 & 1 & \\ & 2 & 1 & 0 & 2 & 1 & \\ & 3 & 0 & 2 & 0 & 0 & = M \\ \text{left} & 4 & 1 & 1 & 0 & 0 & \end{array}$$

$$M^2 = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 6 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

2)

$$M^3 = \begin{bmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{bmatrix}$$

check: 7



$$\begin{array}{cc|cccc} & & & & & \text{right} \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 1 & 0 & 1 \\ & 2 & 1 & 0 & 2 & 1 \\ & 3 & 0 & 2 & 0 & 0 \\ \text{left} & 4 & 1 & 1 & 0 & 0 \end{array}$$

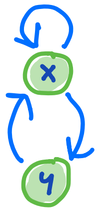
$$\begin{array}{cc|cccc} & & & & & \text{left} \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 1 & 0 & 1 \\ & 2 & 1 & 0 & 2 & 1 \\ & 3 & 0 & 2 & 0 & 0 \\ \text{right} & 4 & 1 & 1 & 0 & 0 \end{array}$$

$$\begin{array}{cc|cccc} & & & & & \text{end} \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 1 & 0 & 1 \\ & 2 & 1 & 0 & 2 & 1 \\ & 3 & 0 & 2 & 0 & 0 \\ \text{start} & 4 & 1 & 1 & 0 & 0 \end{array}$$

$$\begin{array}{cc|cccc} & & & & & \text{start} \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 0 & 1 & 0 & 1 \\ & 2 & 1 & 0 & 2 & 1 \\ & 3 & 0 & 2 & 0 & 0 \\ \text{end} & 4 & 1 & 1 & 0 & 0 \end{array}$$

row, column
convention
is arbitrary
(and same
if symmetric)

Generating functions



$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = M$$

$$M^n_{ij} = (i,j) \text{ entry of } n^{th} \text{ power of } M$$
$$= \#\text{ paths of length } n \text{ from } i \text{ to } j$$

$$\boxed{g_{ij}(t) = \sum_{n=0}^{\infty} M^n_{ij} t^n}$$

generating function, all n at once

Divide into cases:

$$g_{ij}(t) = \underset{\substack{1 \text{ if } i=j\\ 0 \text{ if } i\neq j\\ (0 \text{ steps})}}{\delta_{ij}} + \underbrace{t\, g_{i1}(t) M_{1j}}_{\substack{\text{get to } 1\\ \text{step } 1 \text{ to } j}} + \underbrace{t\, g_{i2}(t) M_{2j}}_{\substack{\text{get to } 2\\ \text{step } 2 \text{ to } j}}$$

$f: \mathbb{N} \to \mathbb{Z}$

$f(n)$ counts something

$$\sum_{n=0}^{\infty} f(n) t^n$$

$$\underset{G}{\begin{bmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}} = \underset{I}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} + t \underset{G}{\begin{bmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}} \underset{M}{\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}}$$

$$G = I + tGM$$
$$GI - tGM = I$$
$$G(I - Mt) = I$$
$$G = (I - Mt)^{-1}$$

inverse matrix

Or sum geometric series

$$g = \sum_{n=0}^{\infty} a^n \Rightarrow g = \frac{1}{1-a}$$

$$G = \sum_{n=0}^{\infty} M^n t^n \Rightarrow G = (I - Mt)^{-1}$$

works for matrices

$$A = \begin{bmatrix} a & b\\ c & d\end{bmatrix} \Longrightarrow A^{-1} = \begin{bmatrix} d & -b\\ -c & a\end{bmatrix} \Big/ (ad-bc)$$

$$I - Mt = \begin{bmatrix} 1-t & -t\\ -t & 1\end{bmatrix}$$

$$\boxed{(I-Mt)^{-1} = \begin{bmatrix} 1 & t\\ t & 1-t\end{bmatrix} \Big/ (1-t-t^2)}$$

How can we understand $\frac{1}{1-t-t^2}$?

$$(\,1 + \square t + \square t^2 + \square t^3 + \square t^4 + \ldots\,)(1-t-t^2) = 1$$

figure out step by step as recurrence relation

$$\begin{array}{r|lllllll}
1 & 1+ & 1t+ & 2t^2+ & 3t^3+ & 5t^4+ & \ldots & \\
-t & & -1t & -1t^2 & -2t^3 & -3t^4 & -5t^5+ & \ldots \\
-t^2 & & & -1t^2 & -1t^3 & -2t^4 & -3t^5 & -5t^6+ \ldots \\
\hline
 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Each term is sum of previous two. **Fibonacci sequence**

n	0	1	2	3	4	5	6	...
m^n_{11}	1	1	2	3	5	8	13	...

1 1 ...

1 1

$$-t^2 \; -t \; +1 \; = 0$$

$$1-t-t^2=0$$

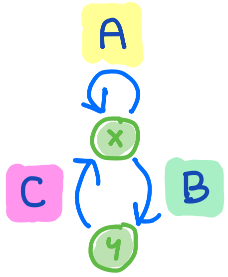
now $\to$ $1 = t + t^2$

reach back 1 step

reach back 2 steps

How can we understand $\frac{1-t}{1-t-t^2}$?

n	0	1	2	3	4	5	6	...	
$1/1-t-t^2$	1	1	2	3	5	8	13	...	
$-t/1-t-t^2$		1	1	2	3	5	8	13	...
	1	0	1	1	2	3	5	...	



Huh? It sure looks like

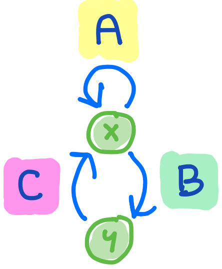
$$\boxed{\frac{1-t}{1-t-t^2} = 1 + \frac{t^2}{1-t-t^2}}$$

$$\frac{1-t-\cancel{t^2} + \cancel{t^2}}{1-t-t^2}$$

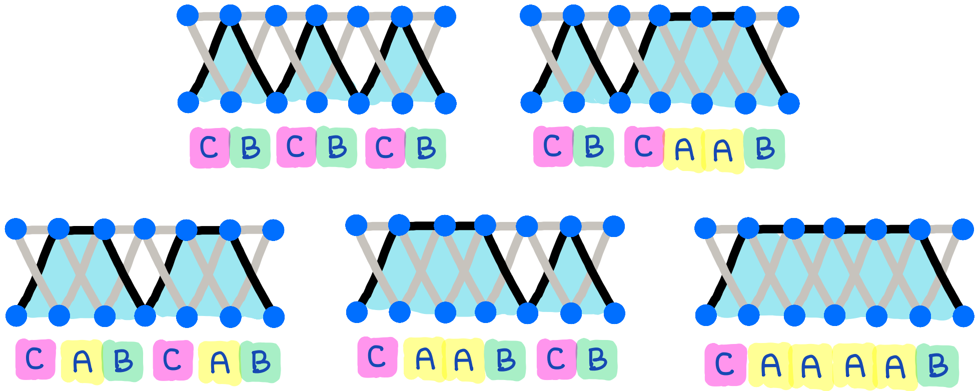
☑ yes!

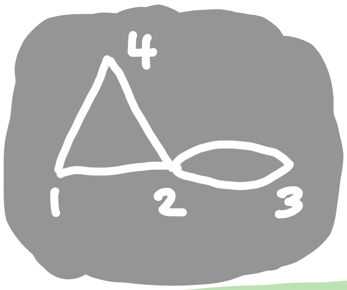
explain

Check: $n=6$, y to y: 5 paths



Adopt left to right convention from probability, CS $\longrightarrow$





$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad I - Mt = \begin{bmatrix} 1 & -t & 0 & -t \\ -t & 1 & -2t & -t \\ 0 & -2t & 1 & 0 \\ -t & -t & 0 & 1 \end{bmatrix}$$

3) $$(I-Mt)^{-1} = \frac{1}{1-7t^2-2t^3+4t^4} \begin{pmatrix} 1-5t^2 & t+t^2 & 2t^2+2t^3 & t+t^2-4t^3 \\ t+t^2 & 1-t^2 & 2t-2t^3 & t+t^2 \\ 2t^2+2t^3 & 2t-2t^3 & 1-3t^2-2t^3 & 2t^2+2t^3 \\ t+t^2-4t^3 & t+t^2 & 2t^2+2t^3 & 1-5t^2 \end{pmatrix}$$

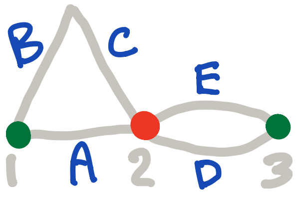
4) $$(I-Mt)^{-1}_{13} = \frac{2t^2+2t^3}{1-7t^2-2t^3+4t^4}$$

Using Mathematica
(Can be done by hand)

Check: $1-7t^2-2t^3+4t^4=0$
$1 = 7t^2+2t^3-4t^4$

n	0	1	2	3	4	5	6			
$1/(1-7t^2-2t^3+4t^4)$	1	0	7	2	45	28	...			
	-4	2	7							
$2t^2$			2	0	14	4	90	56	...	
$+2t^3$				2	0	14	4	90	56	...
	0	0	2	2	14	18	94	146	...	

☑



A|A B C|D
A|C B A|D
B B B C|D

B C|A A|D
B C|C C|D
B C|D D|D

B C|D E|D
B C|E D|D
B C|E E|D

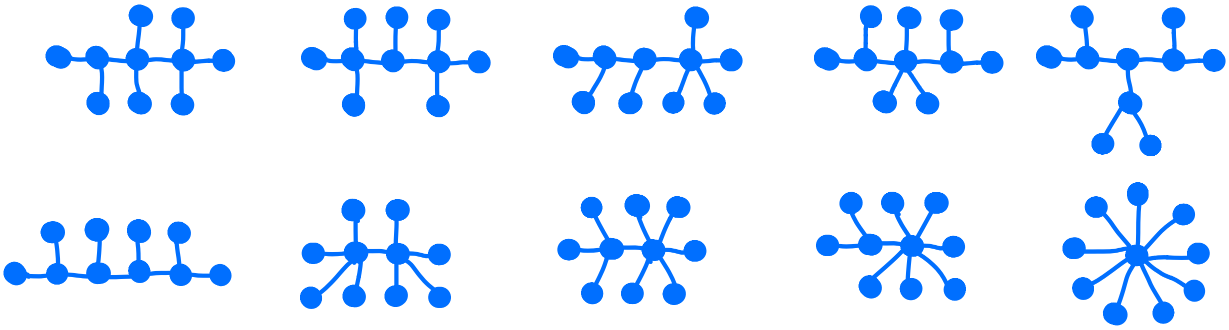
A|A B C|E
A|C B A|E
B B B C|E

B C|A A|E
B C|C C|E
B C|D D|E

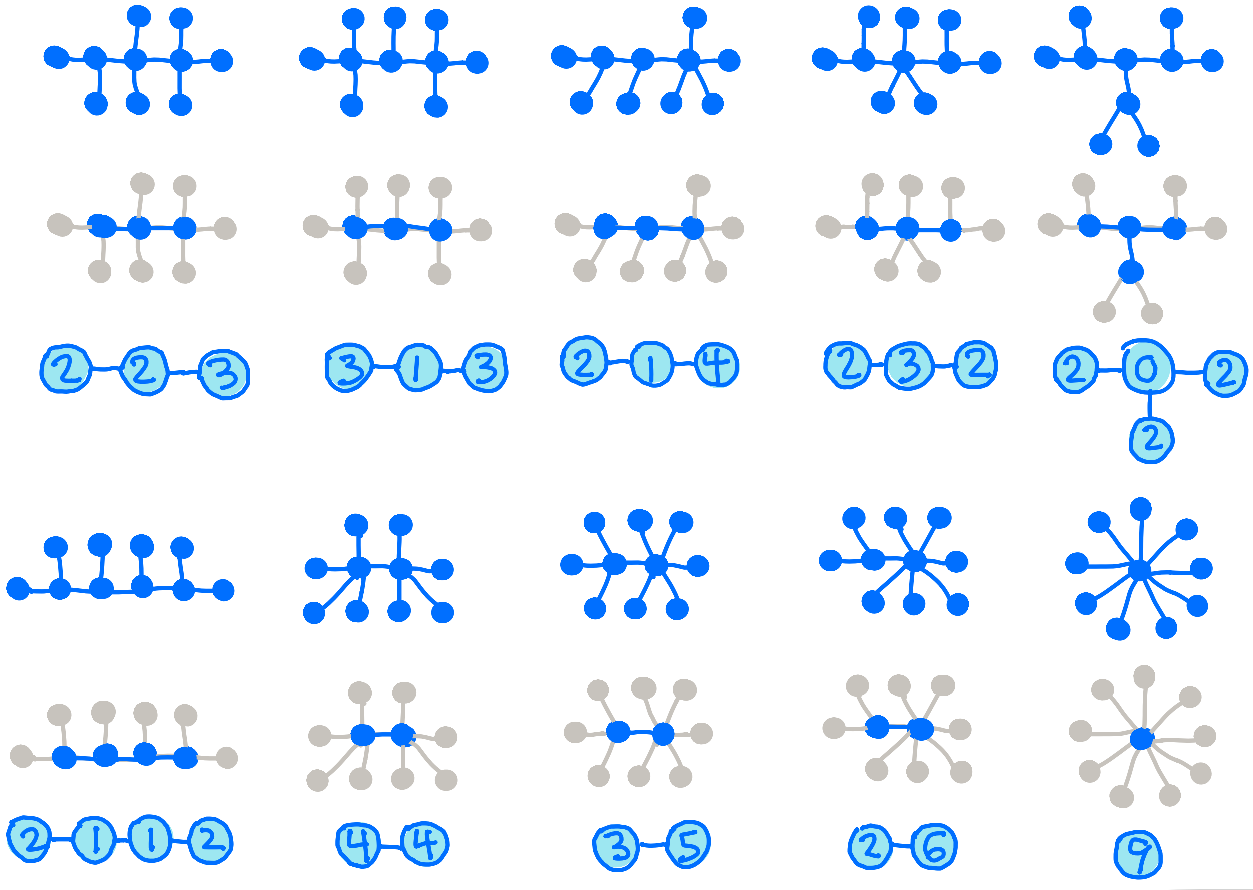
B C|D E|E
B C|E D|E
B C|E E|E

Bonus: 2nd problem in Film is actually easier

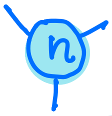
Draw all trees on 10 nodes up to symmetry (no nodes of degree 2)



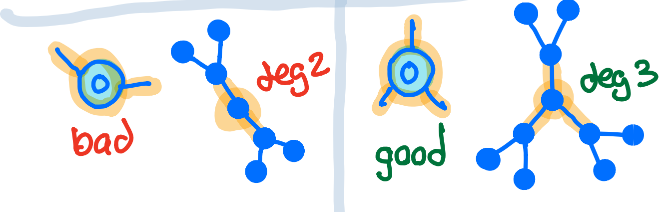
How can we make the drawings easier? Imply "leaves"

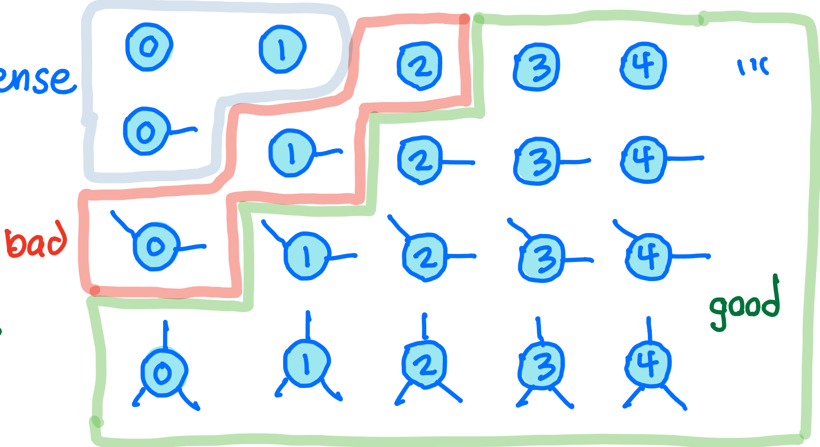


Leaves are implied.
Revised degree rule:



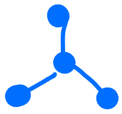
n + edges ≥ 3





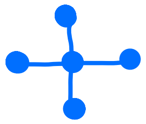
4 nodes : Non-leaves + numbers sum to 4

1 non-leaf : ③



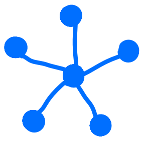
5 nodes : Non-leaves + numbers sum to 5

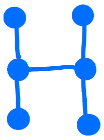
1 non-leaf : ④



6 nodes : 1 non-leaf : ⑤

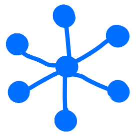
2 non-leaves : ②-②

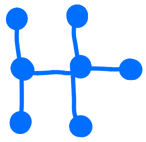




7 nodes : 1 non-leaf : ⑥

2 non-leaves : ②-③

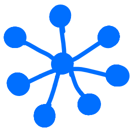


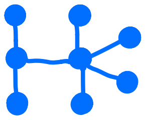


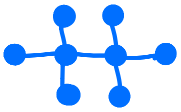
8 nodes : 1 non-leaf : ⑦

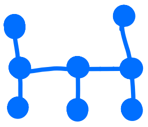
2 non-leaves : ②-④ ③-③

3 non-leaves : ②-①-②









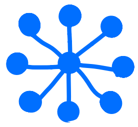
9 nodes : 1 non-leaf : ⑧

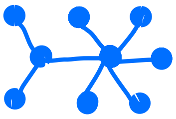
2 non-leaves : ②-⑤

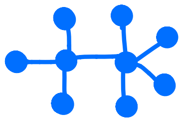
③-④

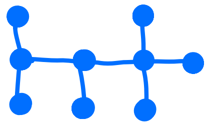
3 non-leaves : ②-①-③

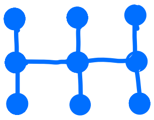
②-②-②









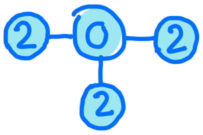


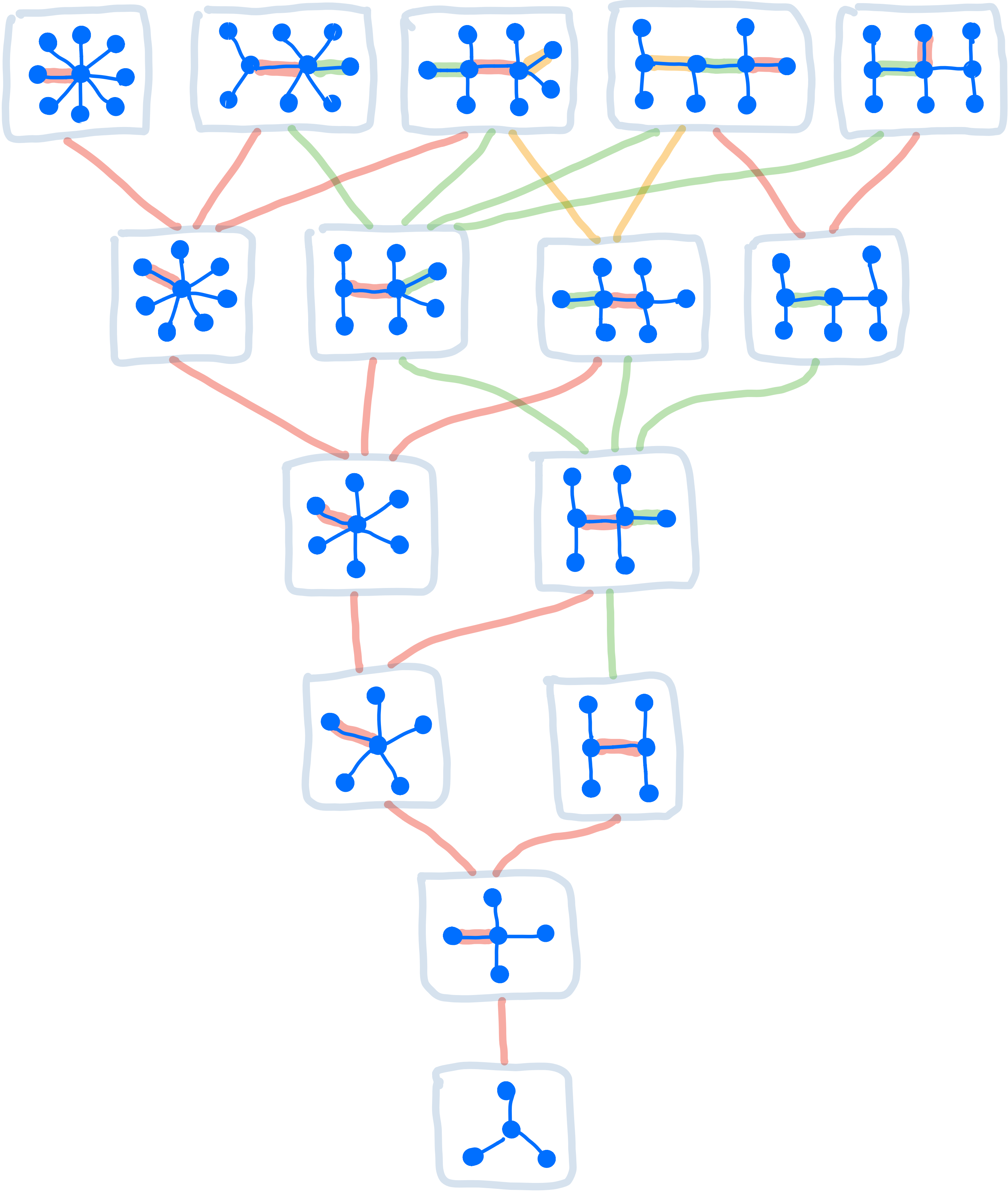
10 nodes : 1 non-leaf : ⑨

2 non-leaves : ②-⑥ ③-⑤ ④-④

3 non-leaves : ②-①-④ ②-②-③ ②-③-② ③-①-③

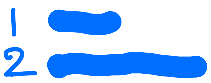
4 non-leaves : ②-①-①-②



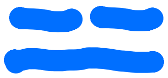


Partially ordered set induced by contracting edges

Some edge contractions are not allowed, create degree 2 nodes.

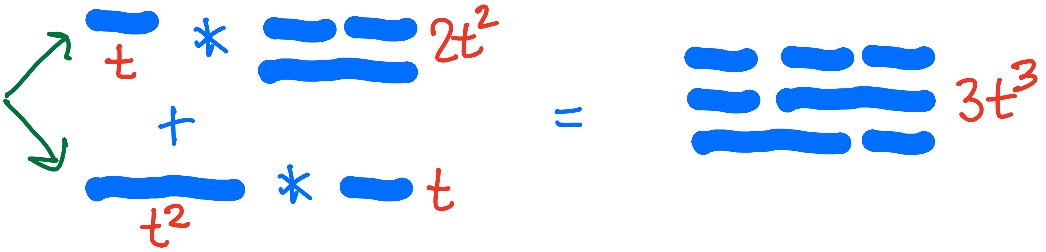


tube length

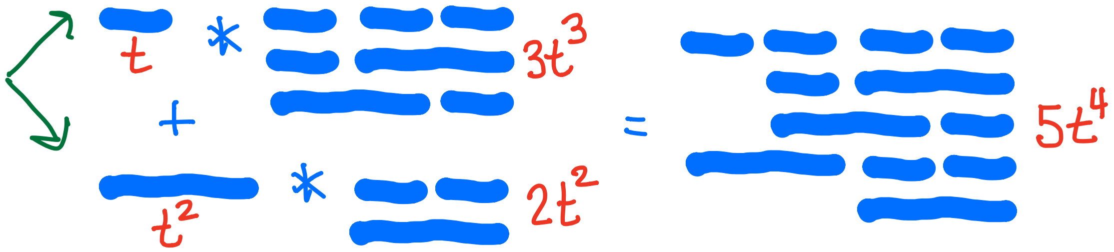
n	f(n)	list
0	1	1
1	1	 t
2	2	 $2t^2$

After class:
Fibonacci sequence as
sticks length 1 or 2
filling tube of length n
recurrence,
and generating function

$f(3) = f(2) + f(1)$ 3 3



$f(4) = f(3) + f(2)$ 4 5



$$g(t) = \sum_{n=0}^{\infty} f(n)\,t^n = 1 + \ldots$$

$$\boxed{g(t) = 1 + t\,g(t) + t^2 g(t)} \Rightarrow (1-t-t^2)\,g(t) = 1$$

$$\boxed{g(t) = \frac{1}{1-t-t^2}}$$

$$\begin{aligned}
1 &= 1 + \\
+\,t &= 1\cdot t \\
+\,2t^2 &= +\,t\cdot t \quad 1\cdot t^2 \\
+\,3t^3 &= +\,2t^2\cdot t \quad +\,t\cdot t^2 \\
+\,5t^4 &= +\,3t^3\cdot t \quad +\,2t^2\cdot t^2 \\
\vdots &\quad \vdots
\end{aligned}$$

t shifts by 1
t^2 shifts by 2