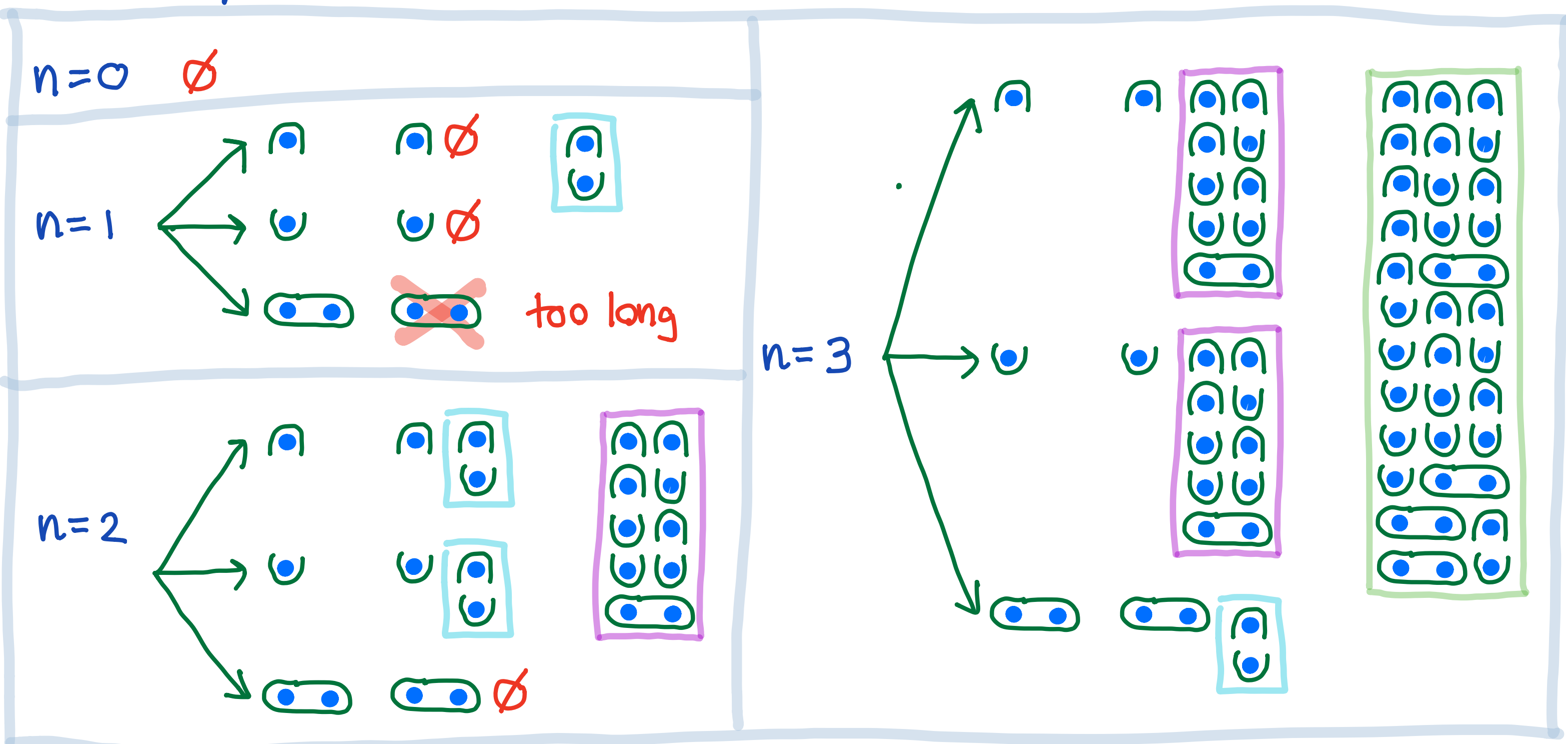
How many words of length n can be formed from ?



$$f(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 2f(n-1)+f(n-2), & n>0 \end{cases}$$

We want to write this without cases, and have it make sense.

$$\boxed{f(n) = 1_0 + 2f(n-1) + f(n-2)}$$

n	0	1	2	3	4	5
1_0	1	0	0	0	0	0
$2f(n-1)$		2 1	2 2	2 5	2 12	2 29
$f(n-2)$			1	2	5	12
$f(n)$	1	2	5	12	29	70

To convert to algebra, use powers of t to record table position

$$\boxed{g(t) = \sum_{n=0}^{\infty} f(n)t^n}$$

generating function

1	1_0	1
$2tg(t)$	$2f(n-1)$	$2t(1+2t+5t^2+12t^3+29t^4+\dots)$
$t^2g(t)$	$f(n-2)$	$t^2(1+2t+5t^2+12t^3+\dots)$
$g(t)$	$f(n)$	$1+2t+5t^2+12t^3+29t^4+70t^5+\dots$

$$f(n) = 1_0 + 2f(n-1) + f(n-2)$$
$$g(t) = 1 + 2tg(t) + t^2g(t)$$

into generating function

$$g(t) = \sum_{n=0}^{\infty} f(n)t^n$$

$$\Downarrow$$

$$g(t) - 2tg(t) - t^2g(t) = 1$$
$$g(t)(1-2t-t^2) = 1$$

learn to read same way

$$\boxed{g(t) = \frac{1}{1-2t-t^2}}$$

*	1	$+2t$	$+5t^2$	$+12t^3$	$+29t^4$	$+70t^5+\dots$
1	1	$+2t$	$+5t^2$	$+12t^3$	$+29t^4$	$+70t^5+\dots$
$-2t$		$-2t(1$	$+2t$	$+5t^2$	$+12t^3$	$+29t^4+\dots)$
$-t^2$			$-t^2(1$	$+2t$	$+5t^2$	$+12t^3+\dots)$
	1	0	0	0	0	0

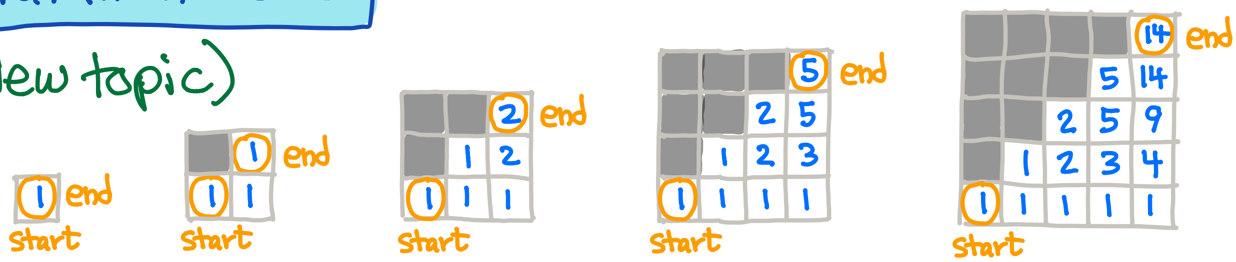
Same calculation more concisely

n	0	1	2	3	4	5
$f(n)$	1	2	5	12	29	70
		$-t^2$	$-2t$	1		

sliding rule for recurrence

Catalan numbers

(New topic)

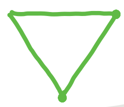


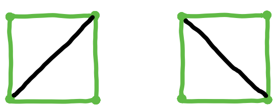
$\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{5}$ $\boxed{14}$

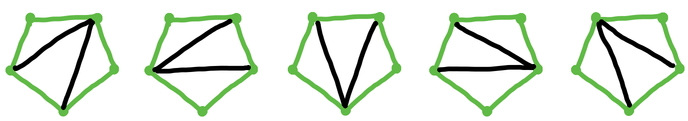
How many lattice paths stay on or below the diagonal?

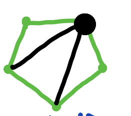
How many ways can we triangulate an n-gon?

$n=2$ $\boxed{1}$ (The empty case, we'll see)

$n=3$ $\boxed{1}$ 

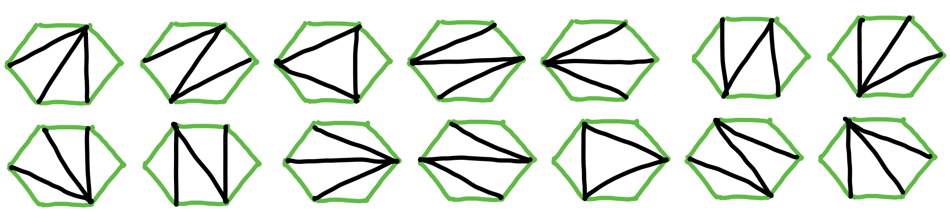
$n=4$ $\boxed{2}$ 

$n=5$ $\boxed{5}$ 



Pick a corner, make a wishbone

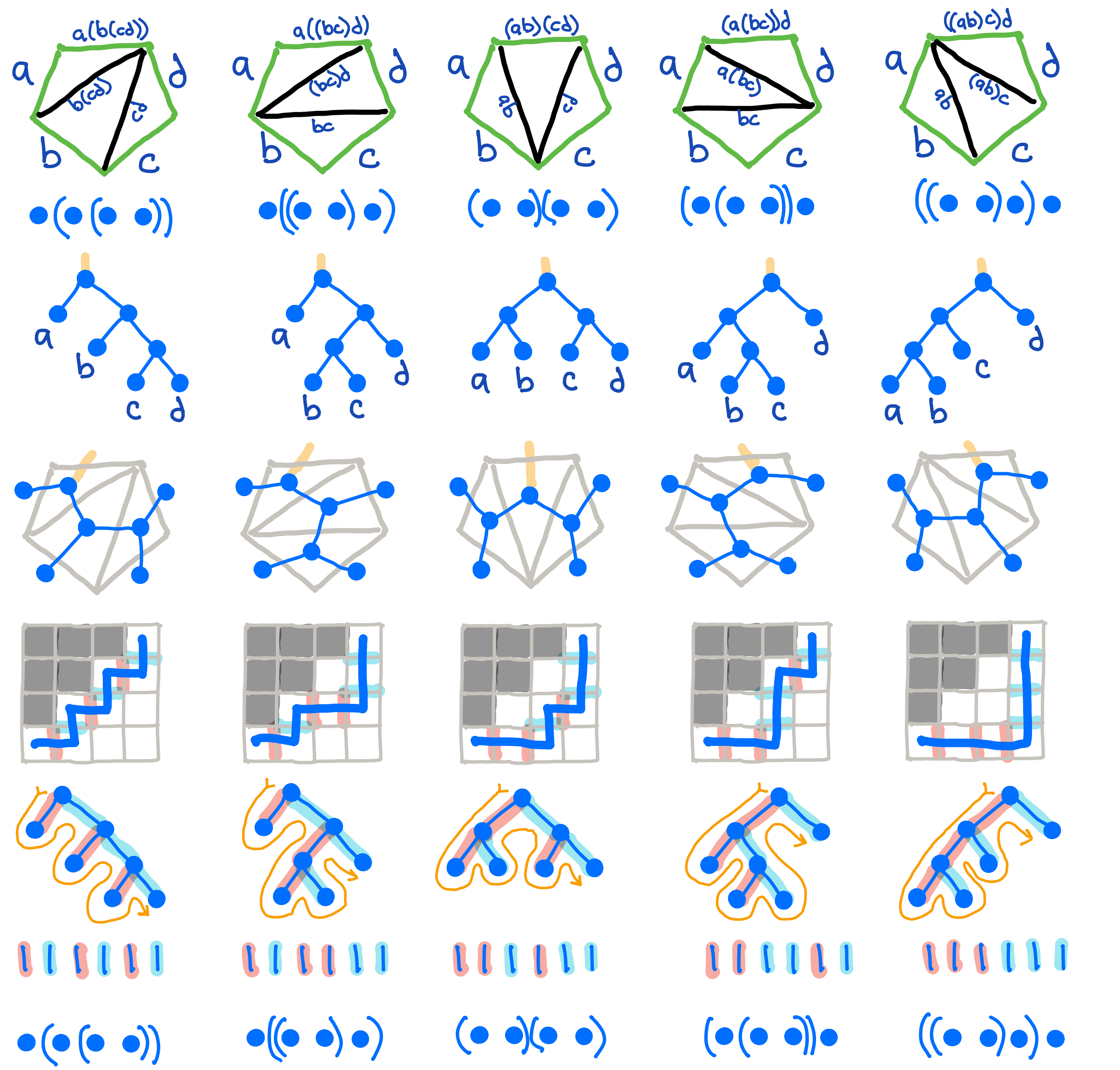
(5 rotations)

$n=6$ $\boxed{14}$ 

(6 rotations) (3 rotations) (3 rotations) (2 rotations)

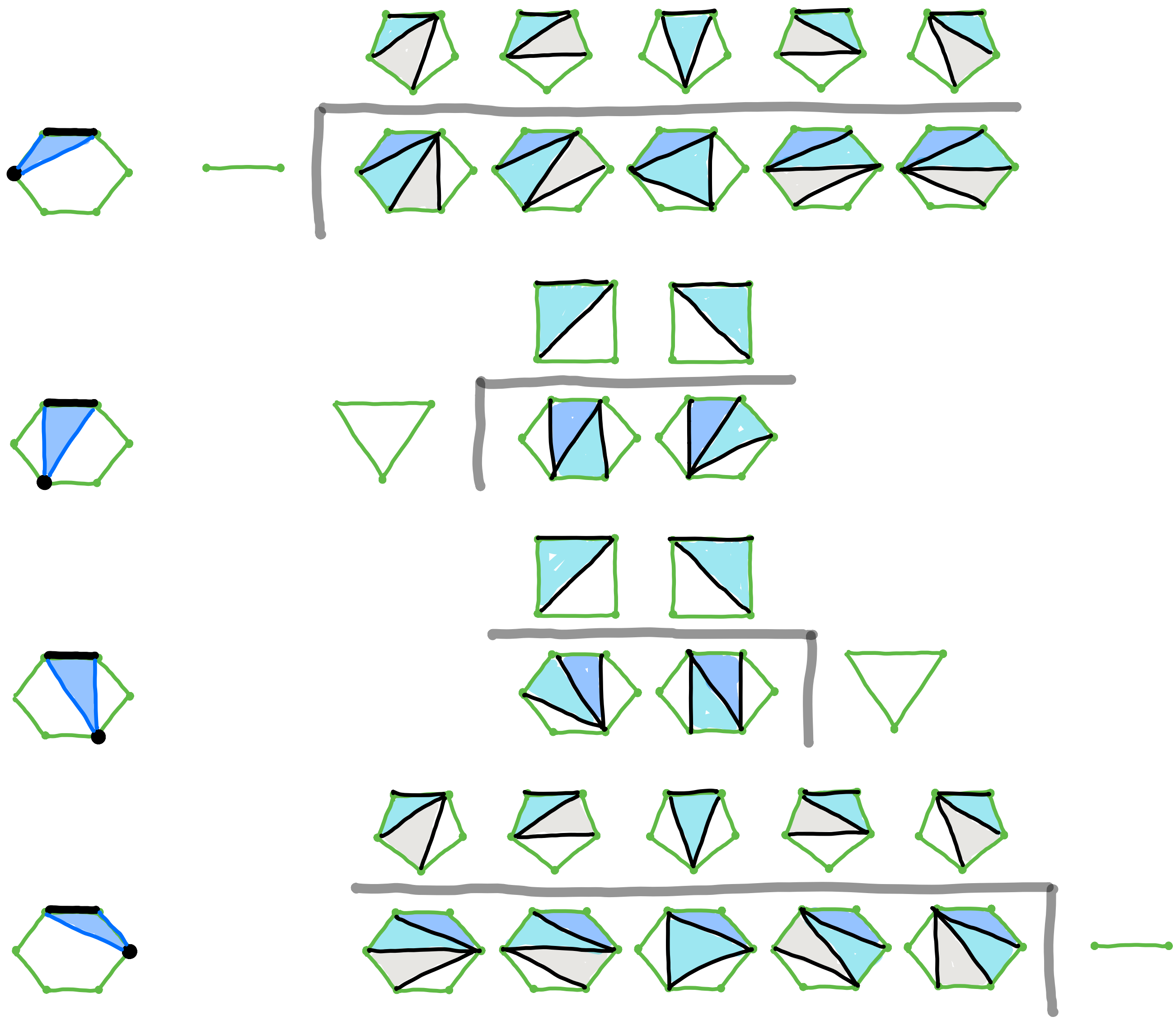
Associative law: How many ways can we parenthesize n terms?

n=1	1	•	No work to do
n=2	1	• •	Only one way to combine terms
n=3	2	(• •)• •(• •)	
n=4	5	•(•(• •)) •((• •)•) (• •)(• •) (•(• •))• ((• •)•)•	
n=5	14	•(•(•(• •))) •(•((• •)•)) •((• •)(• •)) •((•(• •))•) •(((• •)•)•) (• •)(•(• •)) (• •)((• •)•) (•(• •))(• •) ((• •)•)(• •) (•(•(• •)))• (•((• •)•))• ((• •)(• •))• ((•(• •))•)• (((• •)•)•)•	



What is the common pattern? The recurrence?

Each step depends on all previous steps.



1	1	2	5	14	42	...

1
1

1

1	1
1	1

1 1

2

1	1	2
2	1	1

2 1 2

5

1	1	2	5
5	2	1	1

5 2 2 5

14

1	1	2	5	14
14	5	2	1	1

14 5 4 5 14

42

Flip numbers so far, take dot product.

How did I actually figure out the parentheses for 5 terms?