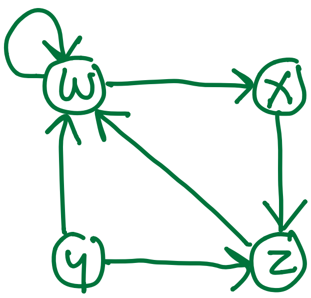
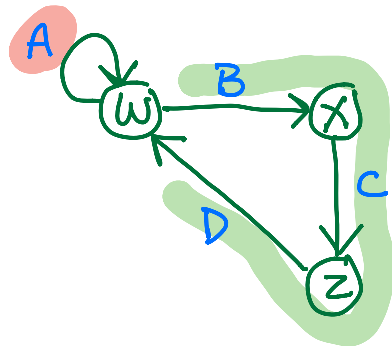
S15 HW1 [2]

Count paths length 10 w to itself



⟹ can't use y



make a list

length	paths
0	∅
1	A
2	AA
3	AAA BCD
4	AAAA ABCD BCDA
5	AAAAA AABCD ABCDA BCDAA

Same problem as filling a tube with sticks of length 1 or 3

$$5 \xrightarrow{A} \text{AAAA} \quad \text{ABCD} \quad \text{BCDA}$$

$$5 \xrightarrow{BCD} \text{AA}$$

let $f(n) = \#$ paths of length n

$\boxed{f(n) = 0 \text{ for } n<0 \qquad f(0) = 1}$

$$f(5) \xrightarrow{A} f(4) \;+\; f(5) \xrightarrow{BCD} f(2)$$

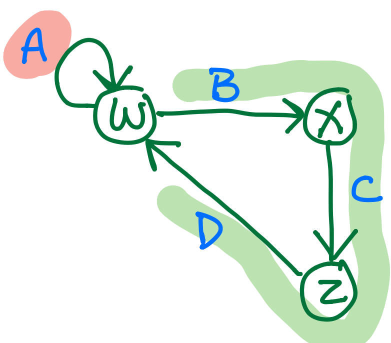
$$f(n) \xrightarrow{A} f(n-1) \;+\; f(n) \xrightarrow{BCD} f(n-3)$$

for $n>0$

n	0	1	2	3	4	5	6	7	8	9	10
f(n)	1	1	1	2	3	4	6	9	13	19	28

$f(n) = f(n-1) + f(n-3)$

check using matrices



$$\begin{array}{c|ccc|l} & w & x & z & \text{out} \\ \hline w & 1 & 1 & & \\ x & & & 1 & \\ z & 1 & & & \\ \text{in} & & & & \end{array} = M \qquad M^{10} \quad (1,1) \text{ entry}$$

$$M = \begin{bmatrix} 1 & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1 & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} 1 & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & & \\ 1 & 1 & \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & & \\ 1 & 1 & \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & & \\ 1 & 1 & \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$M^8 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & \text{[redacted]} \\ 6 & \\ 9 & \end{bmatrix}$$

$$M^{10} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ \text{[redacted]} & & \end{bmatrix}}_{M^2}\underbrace{\begin{bmatrix} 13 & \text{[redacted]} \\ 6 & \\ 9 & \end{bmatrix}}_{M^8} = \begin{bmatrix} 28 & \text{[redacted]} \\ & \end{bmatrix}$$

Generating function:

$$f(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ f(n)=f(n-1)+f(n-3), & n>0 \end{cases}$$

"slang" $\boxed{f(n) = 1_0 + f(n-1) + f(n-3)}$

$$g(t) = \sum_{n=0}^{\infty} f(n) t^n$$

we legitimately can write

$$\boxed{g(t) = 1 + t g(t) + t^3 g(t)}$$

solve: $g(t) - t g(t) - t^3 g(t) = 1$

$$\boxed{g(t) = \frac{1}{1-t-t^3}}$$

"$1-t-t^3=0$" gets used in table

⇓ "$1 = t + t^3$"

⇓ $\boxed{f(n) = f(n-1) + f(n-3)}$

$$h(t) = \frac{t}{1-t-t^3} \qquad h(t) = \sum_{n=0}^{\infty} j(n) t^n$$

$$h(t) = t g(t)$$

$$g(t) = 1 + t + t^2 + 2t^3 + 3t^4 + 4t^5 + 6t^6 + \ldots$$

$$h(t) = 0 + t + t^2 + t^3 + 2t^4 + 3t^5 + 4t^6 + 6t^7 + \ldots$$

what is $j(n)$?

$$(n) = \begin{cases} 0, & n<1 \\ 1, & n=1 \\ j(n)=j(n-1)+j(n-3), & n>1 \end{cases}$$

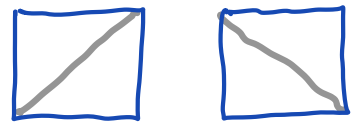
S 2014 Exam 1 [5]

How many ways to cut an 8-gon into 3 pieces

$n=8$ $\quad c=2$ cuts

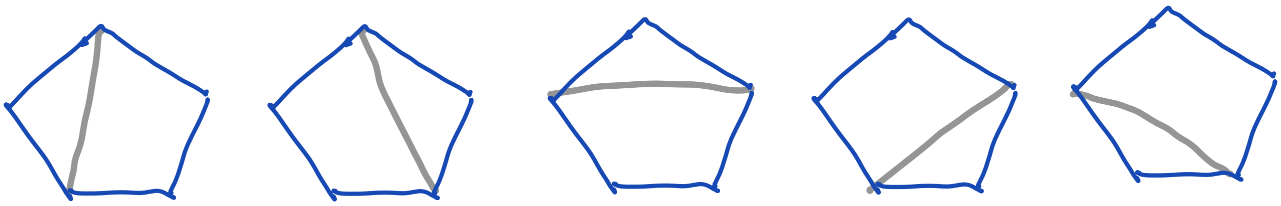
		n = 3	4	5	6	7	8
	0	1	1	1	1	1	1
	1	0	2	5	9	14	20
c	2	0	0				

$n=4$



$n=5$ 1 cut

formula: $\binom{n}{2}$ edges − n boundary edges



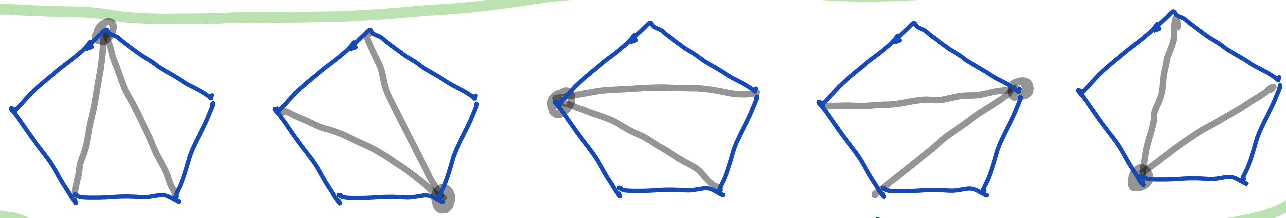
$n=5 \quad \binom{5}{2}-5 = 10-5 = 5$ ✓

$n=6 \quad \binom{6}{2}-6 = 15-6 = 9$

$n=7 \quad \binom{7}{2}-7 = 21-7 = 14$

$n=8 \quad \binom{8}{2}-8 = 28-8 = 20$

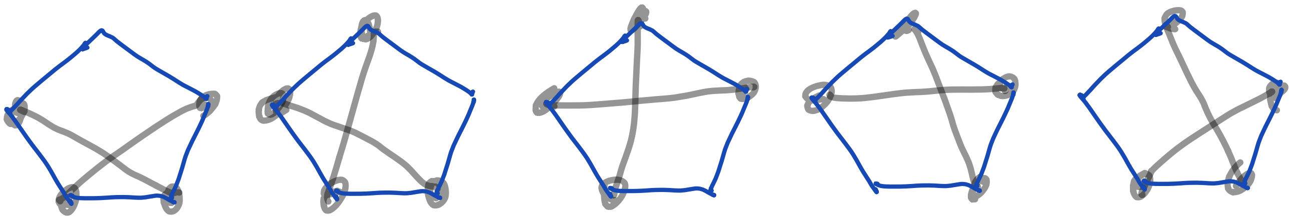
$n=5$ 2 cuts



good

how many pairs of interior edges are there? $\binom{5}{2}=10$

what's wrong with the others? Can we count them?



count bad pairs: $\binom{n}{4}$

Formula for two cuts: $\binom{\binom{n}{2}-n}{2} - \binom{n}{4}$

Check for $n=5$: $\binom{\binom{5}{2}-5}{2} - \binom{5}{4} = \binom{5}{2} - 5 = 10-5=5$

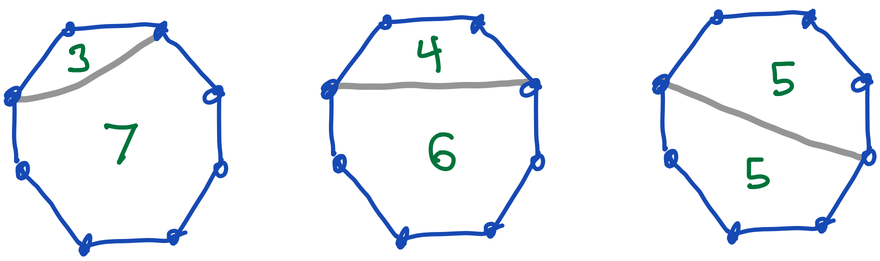
Go for $n=8$: $\binom{\binom{8}{2}-8}{2} - \binom{8}{4}$

$$= \binom{28-8}{2} - \frac{\cancel{8}\cdot 7\cdot \cancel{6}^{\,2}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$$

$$\binom{20}{2} - 70 = 190-70 = 120$$

Alternate approach

Count ordered pairs of cuts
then divide by two
expect $240 \div 2 = 120$



8   8   4

$0+14$ $2+9$ $5+5$

$8\cdot 14 + 8\cdot 11 + 4\cdot 10$

How many?

How to add a second cut?

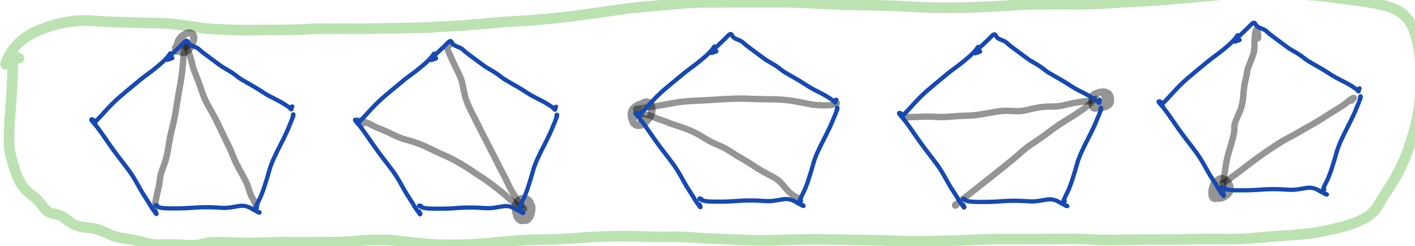
$8\cdot 25 + 40 = 240$
200

$\div 2 = 120$ ✓

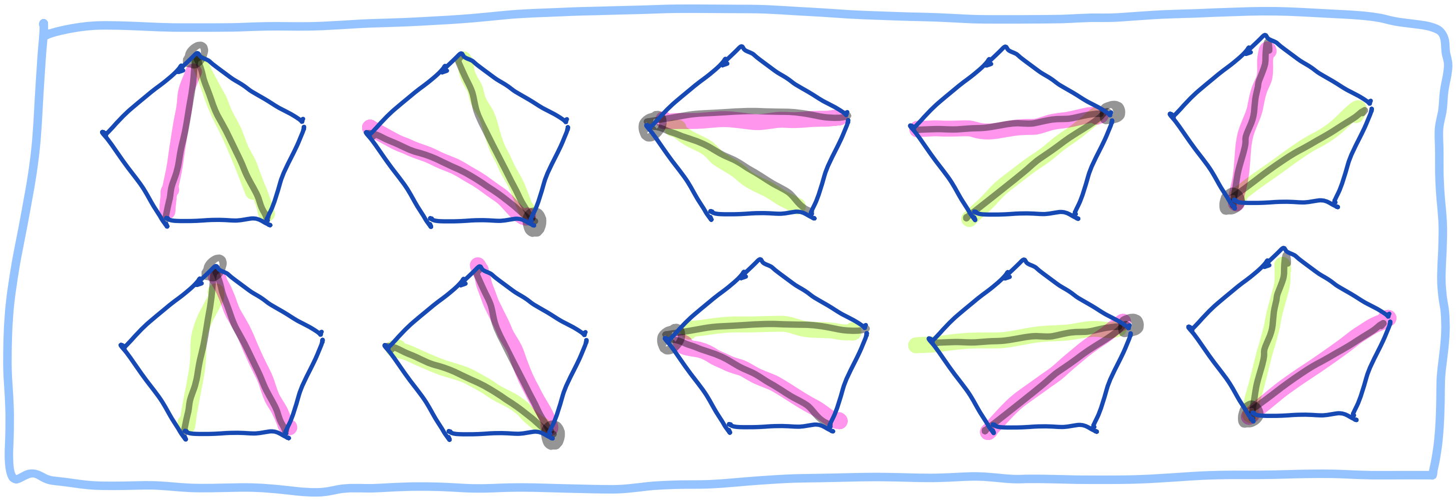
n

	3	4	5	6	7	8
0	1	1	1	1	1	1
1	0	2	5	9	14	20
c 2	0	0				$\boxed{120}$

Why did we divide by two, to go from ordered to unordered cuts?



unordered cuts $n=5$



ordered cuts $n=5$

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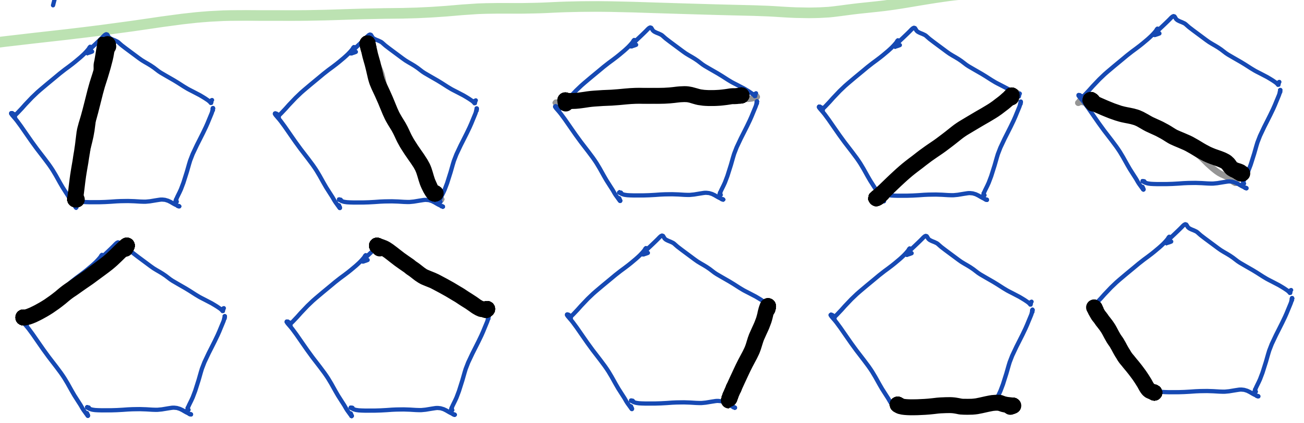
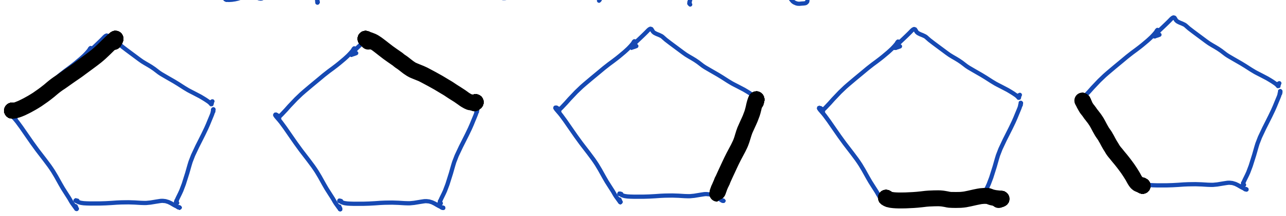
why $\binom{n}{2}-n$ for first cut?



all pairs of vertices = all edges including boundary edges

$\binom{n}{2}$

subtract boundary edges n



For any n-gon, n vertices and n edges

Making change for 20¢ ① ② ③

SIS Makeup test 1 [4]

	0	1	2	3	4	5	6	7	8	9	10	11	12	n cents
①	1	1	1	1	$\boxed{1}$	1	1	1	1	1	1	1	1	
① ②	1	1	$\boxed{2}$	2	$\boxed{3}$	3	4	4	5	5	6	6	7	
① ② ③	1	1	2	3	4	5	7	8	10	12	14	16	$\boxed{19}$	

check

✓ no to all ② coins 0, 1, 2, 3, 4, 5, 6

① ② n cents

use only ① → "① n cents"

use at least one ② → ② and "① ② n-2 cents"

Example n=5:

① ② 5 cents

use only ① → { $\boxed{①①①①①}$ } 1

+

use at least one ② → { ② ① ① ①, ② ② ① } = ② and { ① ① ①, ② ① } 2

① ② ③ n cents

use only ① ② → $\boxed{\text{"① ② n cents"}}$

use at least one ③ → ③ and $\boxed{\text{"① ② ③ n-3 cents"}}$

check 19 ways $n=12$ cents ① ② ③

How many ③'s ?

what's left? finish using only ① ②

how many ③'s			
	0	12¢	7
	1	9¢	5
	2	6¢	4
	3	3¢	2
	4	0¢	1

19 ✓

Generating function approach.

change making same problem as degree 12 monomials

in x, y, z } variable

1 2 3 } degree

① ② ③

$n=5$ ①①①①① $\longleftrightarrow$ x^5

②①①① $\longleftrightarrow$ x^3y

②②① $\longleftrightarrow$ xy^2

To get all possibilities for all n,

$$(1+x+x^2+x^3+\dots)(1+y+y^2+y^3+\dots)(1+z+z^2+\dots)$$

$$= 1 + x + (x^2+y) + (x^3+xy+z) + (x^4+x^2y+y^2+xz) + \dots$$

$$\boxed{x=t \qquad y=t^2 \qquad z=t^3}$$

$$(1+t+t^2+t^3+\dots)(1+t^2+t^4+t^6+\dots)(1+t^3+t^6+\dots)$$

$$= 1 + t + 2t^2 + 3t^3 + 4t^4 + \dots$$

$$= \left(\frac{1}{1-t}\right)\left(\frac{1}{1-t^2}\right)\left(\frac{1}{1-t^3}\right)$$

$$= \frac{1}{1-t-t^2+t^4+t^5-t^6}$$

$$(1-t)(1-t^2) = 1-t-t^2+t^3$$

$$(1-t^3)(1-t-t^2+t^3)$$

$$= \begin{array}{c|cccc} & 1 & -t & -t^2 & t^3 \\ \hline 1 & 1 & -t & -t^2 & \cancel{t^3} \\ -t^3 & \cancel{-t^3} & t^4 & t^5 & -t^6 \end{array}$$

$$= 1-t-t^2+t^4+t^5-t^6$$

translate into recurrence

$$1 = t+t^2-t^4-t^5+t^6$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	n
① ② ③	1	1	2	3	4	5	7	8	10	12	14	16	$\boxed{19}$	
							1	-1	-1		1	1		

7 −8 −10 +14 +16 = 19 ✓

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idea of translating generating function denominator into recurrence relation

$$1-t-t^2+t^4+t^5-t^6 = 0$$

$$\begin{array}{ccccccc} 1 & = & t & +t^2 & -t^4 & -t^5 & +t^6 \\ f(n) & = & f(n-1) & +f(n-2) & -f(n-4) & -f(n-5) & +f(n-6) \end{array}$$