last class

$$f(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 2f(n-1)+f(n-2), & n>0 \end{cases}$$

We want to write this without cases, and have it make sense.

$$\boxed{f(n) = 1_0 + 2f(n-1) + f(n-2)}$$

n	0	1	2	3	4	5
1_0	1	0	0	0	0	0
$2f(n-1)$		2 1	2 2	2 5	2 12	2 29
$f(n-2)$			1	2	5	12
$f(n)$	1	2	5	12	29	70

Haskell

```
Prelude> words = 1 : [ 2*a + b | (a, b) <- zip words (0 : words) ]
Prelude> take 6 words
[1,2,5,12,29,70]
```

lazy evaluation

	1	2	5	12	29	words
0	1	2	5	12		0:words
1,0	2,1	5,2	12,5	29,12		zip
2	5	12	29	70		2a+b

words = 1 2 5 12 29 70 1: list

Catalan numbers

$$C_n = \underset{0}{\boxed{1}}, \underset{1}{\boxed{1}}, \underset{2}{\boxed{2}}, \underset{3}{\boxed{5}}, \underset{4}{\boxed{14}}, \ldots$$

They can be found in Pascal's triangle

$$1\ \boxed{1}$$
$$1 \quad 1$$
$$1 \quad 2\ \boxed{1}\ 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6\ \boxed{2}\ 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20\ \boxed{5}\ 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70\ \boxed{14}\ 56 \quad 28 \quad 8 \quad 1$$

$$\boxed{C_n = \binom{2n}{n} - \binom{2n}{n+1}}$$

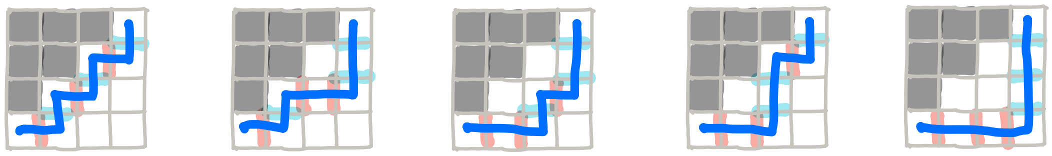
$$\frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} - \frac{2n(2n-1)\cdots(n+1)n}{(n+1)n(n-1)\cdots 1}$$

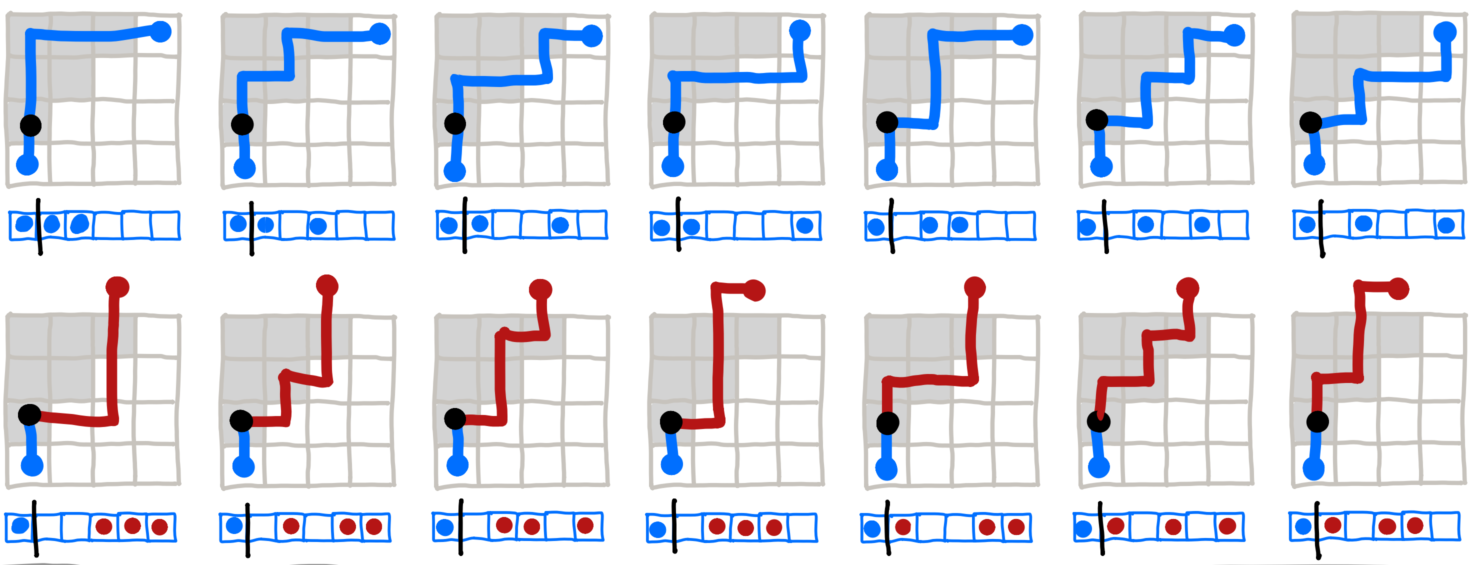
$$\frac{2n(2n-1)\cdots(n+1)(n+1)}{(n+1)n(n-1)\cdots 1} - \frac{2n(2n-1)\cdots(n+1)n}{(n+1)n(n-1)\cdots 1}$$

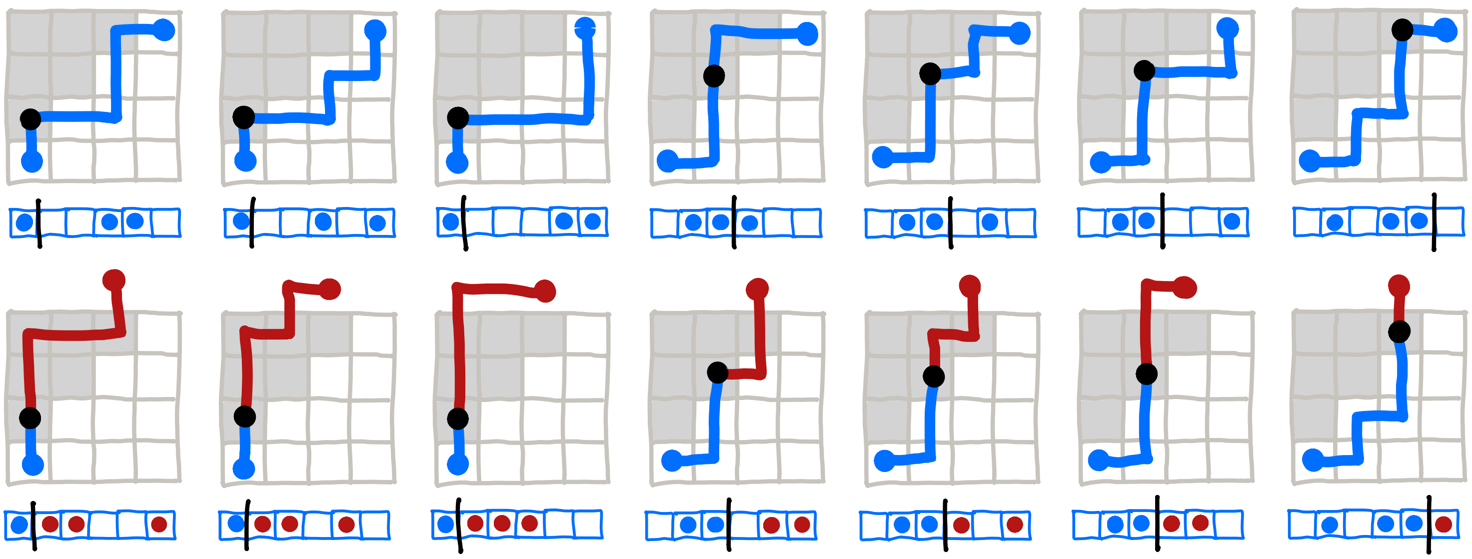
Why? André's reflection method.

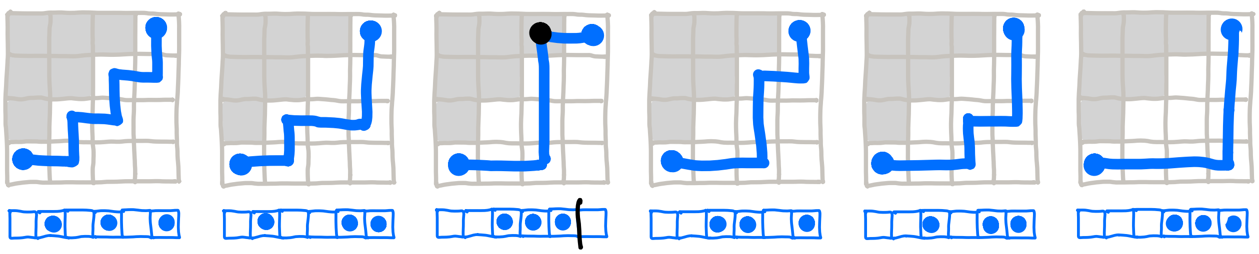
$$\frac{1}{(n+1)}\frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} = \boxed{\frac{1}{n+1}\binom{2n}{n}}$$

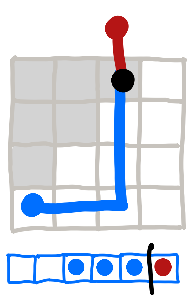
$C_3 = 5$, valid paths on 4x4 grid $\boxed{5} = 20 - 15$











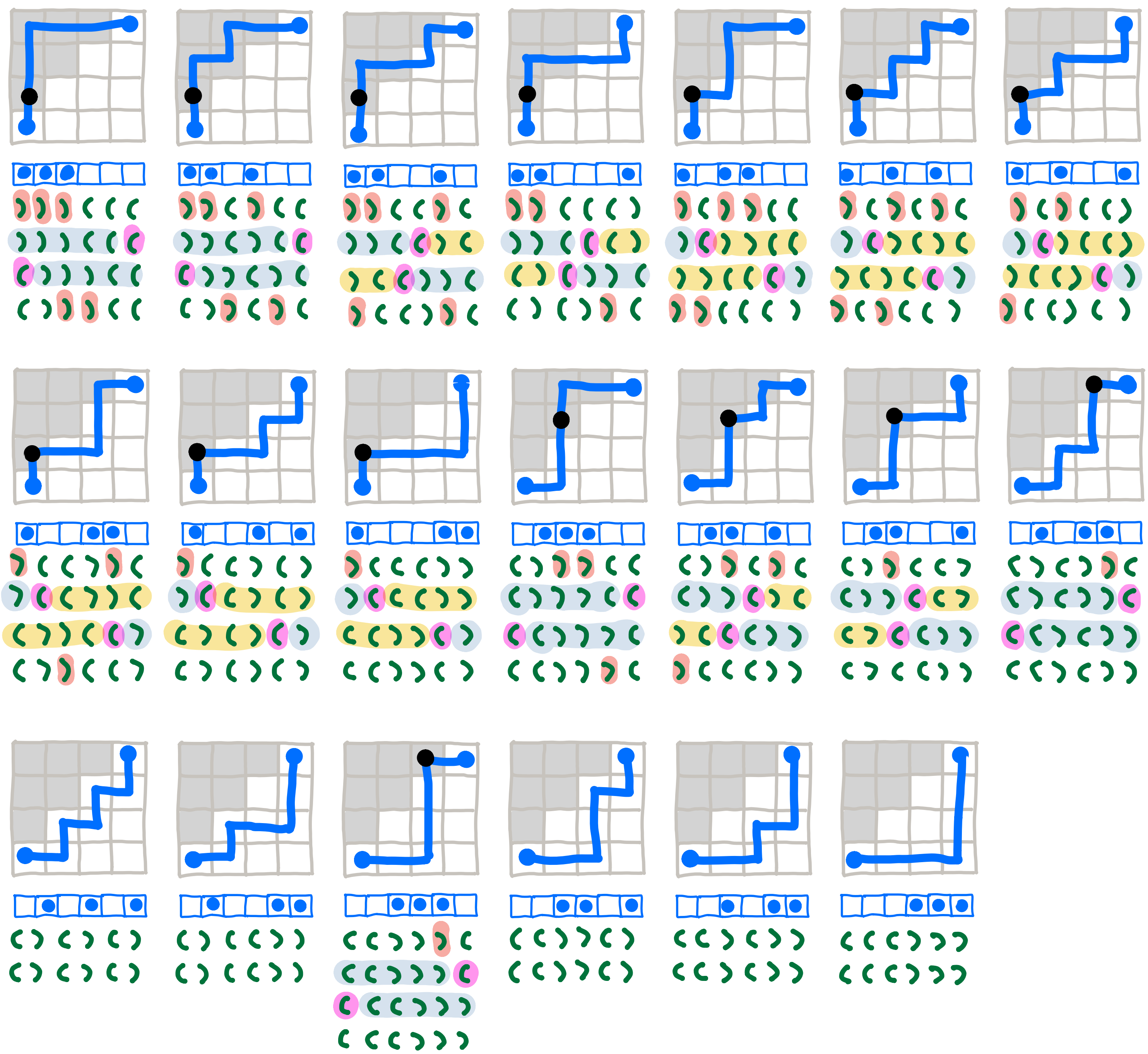
All paths $\binom{6}{3}$ ups

Bad paths flip to $\binom{6}{4}$ ups

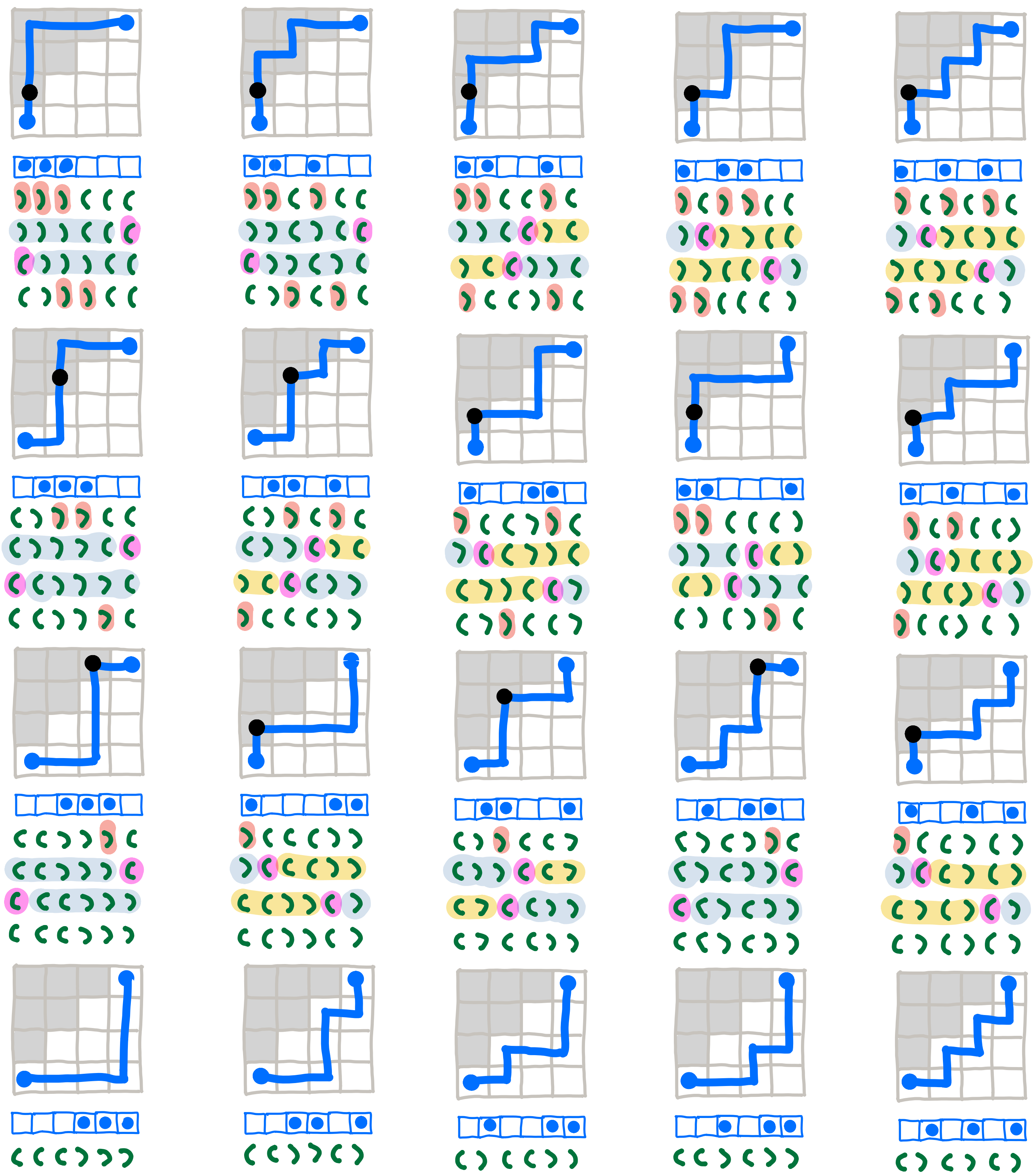
$\binom{6}{3} - \binom{6}{4} = 20 - 15 = \boxed{5}$

Second proof, explain the denominator $\frac{1}{n+1}\binom{2n}{n}$

One of many equal sized groups. Find the others...



Rearrang in strands: $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{4}\binom{6}{3} = \frac{20}{4} = 5$



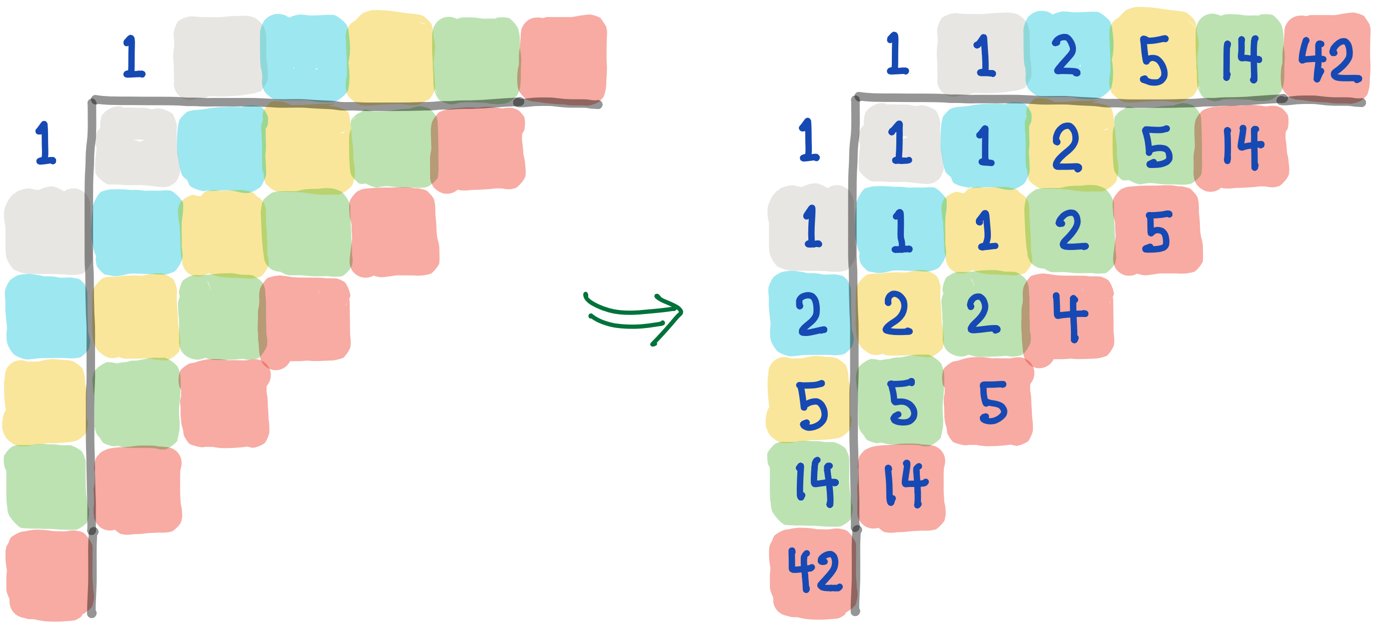
What about generating function?

Play with different ways to present recursion.

From last class: $\boxed{1}\boxed{1}\boxed{2}\boxed{5}\boxed{14}\boxed{42}\ \cdots$

$$\begin{array}{c}1\\1\\\hline 1\end{array}\qquad
\begin{array}{cc}1&1\\1&1\\\hline 1&1\\ &2\end{array}\qquad
\begin{array}{ccc}1&1&2\\2&1&1\\\hline 2&1&2\\ &&5\end{array}\qquad
\begin{array}{cccc}1&1&2&5\\5&2&1&1\\\hline 5&2&2&5\\ &&&14\end{array}\qquad
\begin{array}{ccccc}1&1&2&5&14\\14&5&2&1&1\\\hline 14&5&4&5&14\\ &&&&42\end{array}$$

Flip numbers so far, take dot product.



Let $g(t) = \sum_{n=0}^{\infty} C_n t^n$

Then $g(t) = 1 + t\,g(t)^2$

$t\,g(t)^2 - g(t) + 1 = 0$

$\boxed{ax^2 + bx + c \Rightarrow x = \dfrac{-b \pm \sqrt{b^2-ac}}{2a}}$ $\Rightarrow g(t) = \dfrac{1 \pm \sqrt{1-4t}}{2t}$