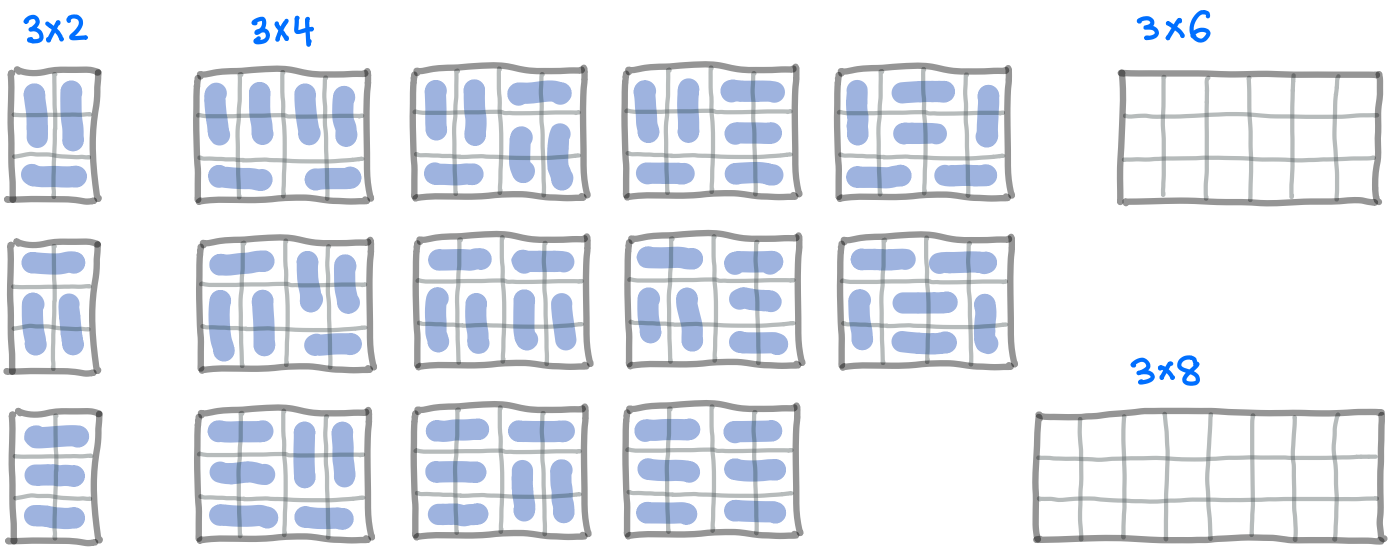
Domino tilings

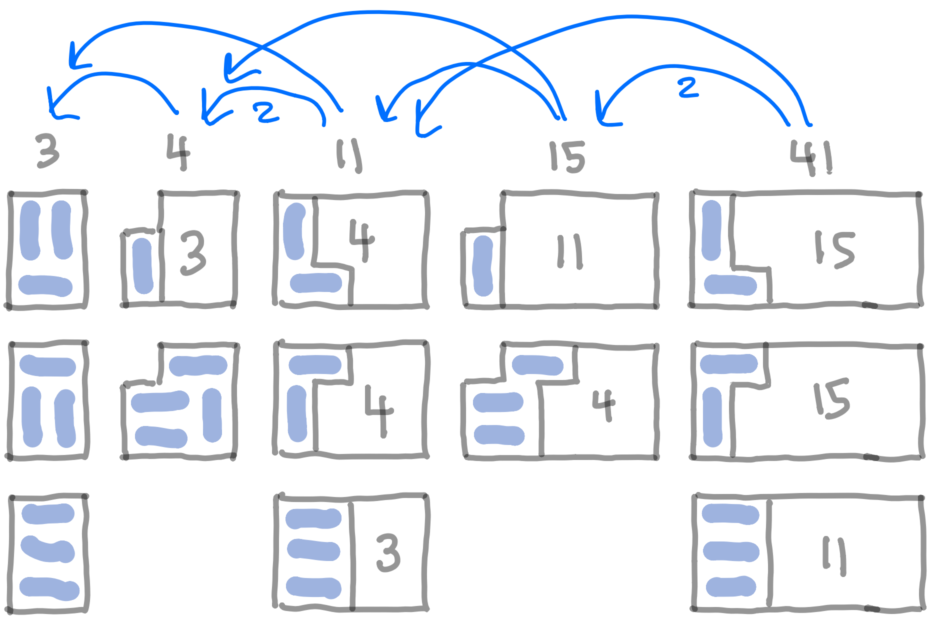
Let $f(n) = \#$ of domino tilings of a $3 \times 2n$ grid.

$f(1) = 3$, $f(2) = 11$

Find $f(3)$, $f(4)$, generating function.

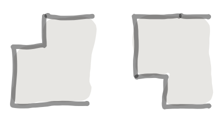


First try (scrap work) break into cases from the left



Use symmetry

Treat

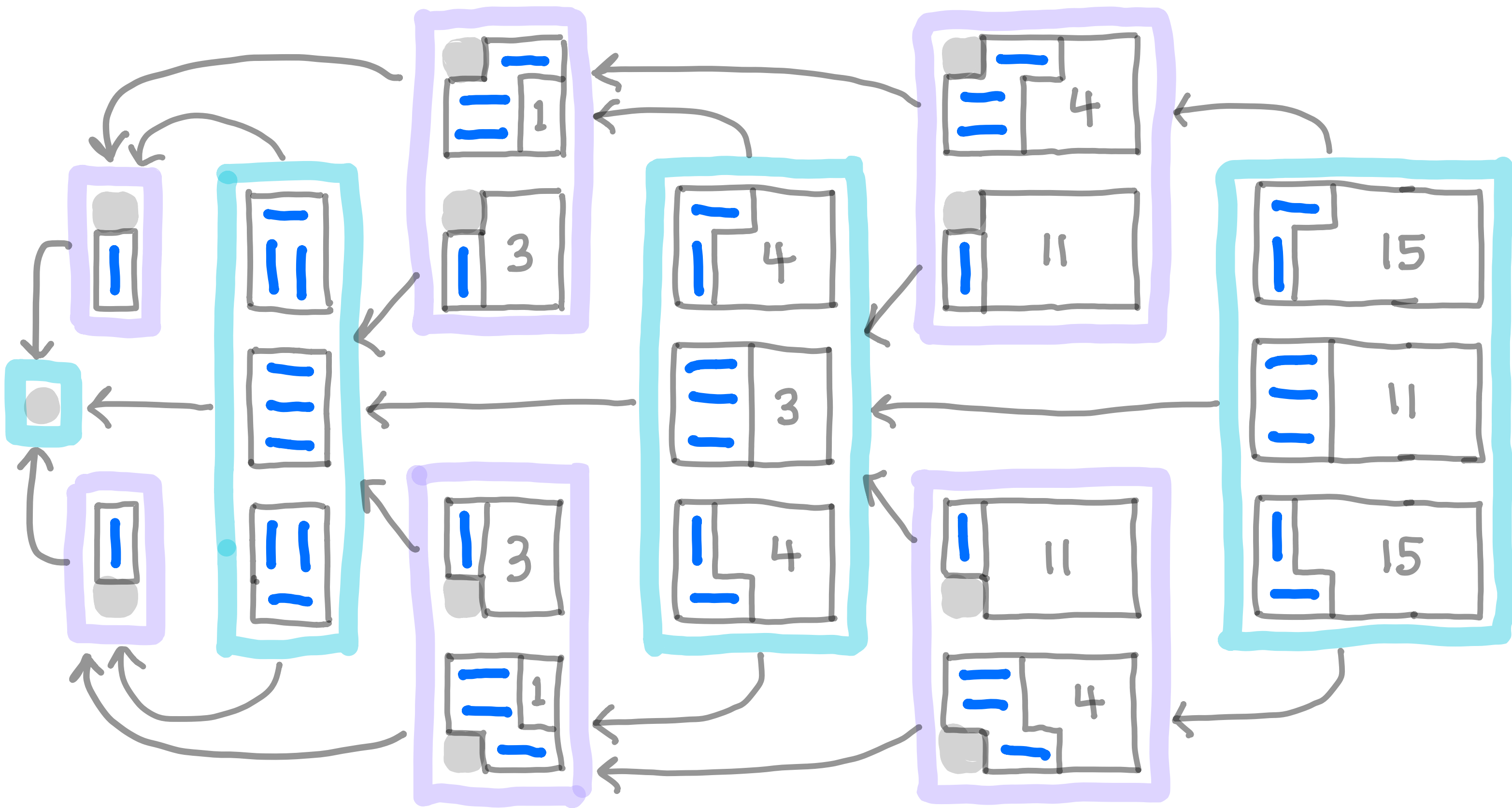


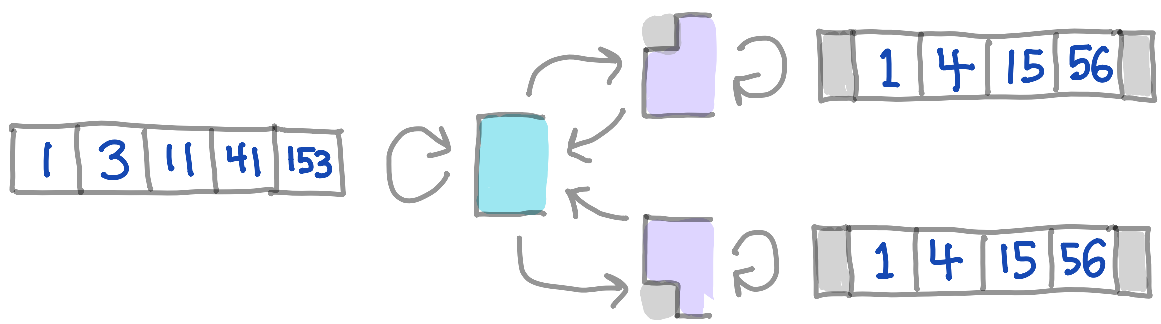
as same case

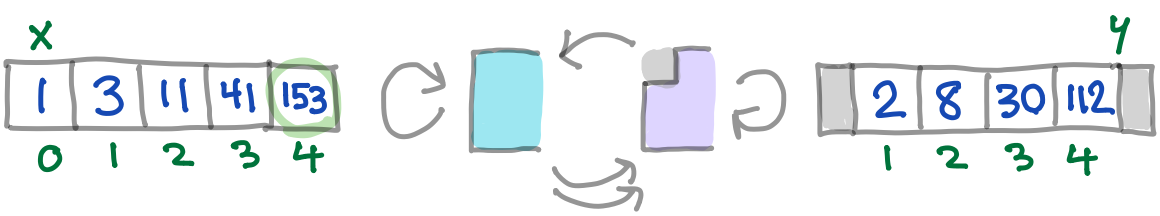
3 4 11 15 41 56 153

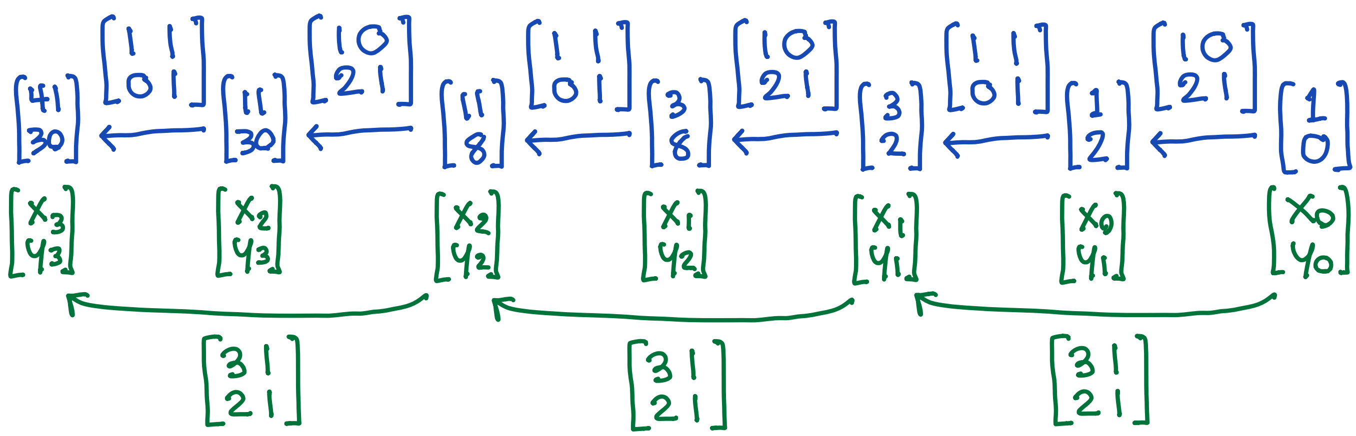
$\boxed{f(3) = 41}$ $\boxed{f(4) = 153}$

Redraw more carefully:









$$\begin{bmatrix}3&1\\2&1\end{bmatrix}^2=\begin{bmatrix}11&4\\8&3\end{bmatrix},\quad \begin{bmatrix}11&4\\8&3\end{bmatrix}^2=\begin{bmatrix}153&56\\112&41\end{bmatrix},\quad \begin{bmatrix}153&56\\112&41\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}153\\112\end{bmatrix}$$

$$\boxed{g(t)=\sum_{n=0}^{\infty} x_n}$$

$$h(t)=\sum_{n=0}^{\infty} y_n$$

$$\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+t\begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{bmatrix}g(t)\\h(t)\end{bmatrix}$$

$$\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-t\begin{bmatrix}3&1\\2&1\end{bmatrix}\right)\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$(1-3t)(1-t)-2t\cdot t = 1-4t+t^2$$

$$\begin{bmatrix}1-3t&-t\\-2t&1-t\end{bmatrix}\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1-3t&-t\\-2t&1-t\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1-t&t\\2t&1-3t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\Big/(1-4t+t^2)=\begin{bmatrix}1-t\\2t\end{bmatrix}\Big/(1-4t+t^2)$$

So $\boxed{g(t)=\dfrac{1-t}{1-4t+t^2}}$

$$1-4t+t^2=0 \Rightarrow 1=4t-t^2$$

n	0	1	2	3	4	...
$1/(1-4t+t^2)$	1	4	15	56	209	...
$-t/(1-4t+t^2)$	0	1	4	15	56	...
x_n	1	3	11	41	153	...

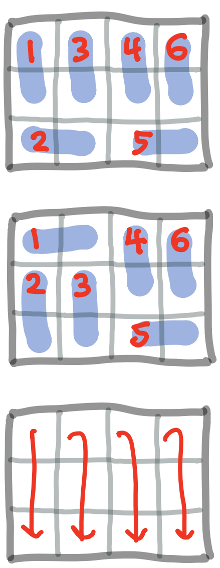
use recurrence $1=4t-t^2$

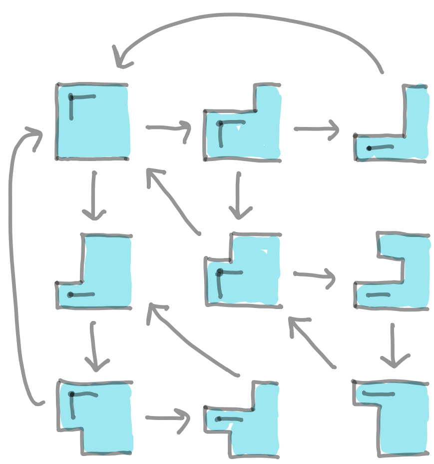
shift by t

OEIS A001835

Second approach: Remove dominos in canonical order.

What does the frontier look like?

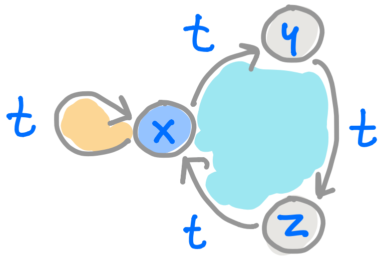




First attempt at graph

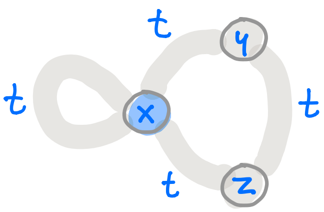
We need a "calculus" for walks on graphs, to get generating function.

Take simpler example: Represent each path by product of edge labels.

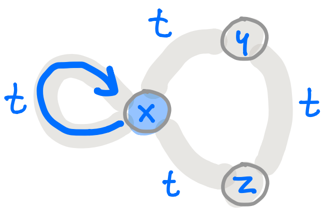


Generating functions are sums of all paths with given start, end vertices.

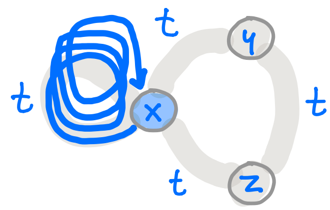
We can simplify graph if it gives same generating function. What are rules?



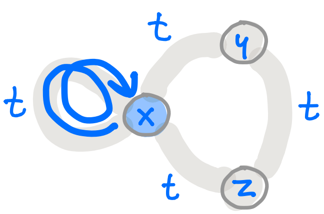
paths x to itself: $\boxed{1+t+t^2+2t^3+3t^4+\dots=\frac{1}{1-(t+t^3)}}$



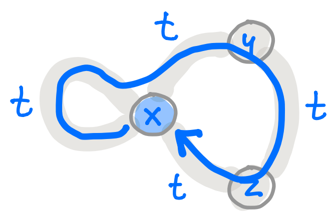
t



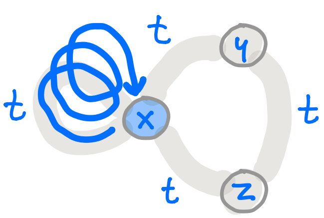
$t\ t\ t\ t$



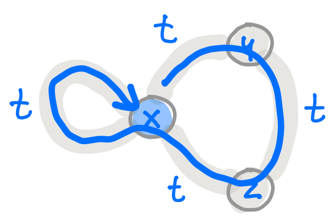
$t\ t$



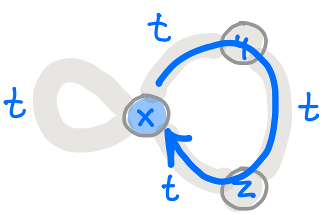
$t\ t^3$



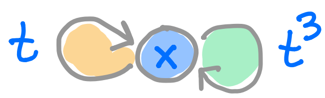
$t\ t\ t$



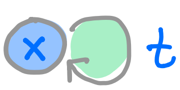
$t^3\ t$



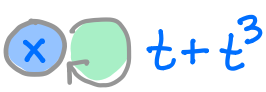
t^3

same: 

same: 



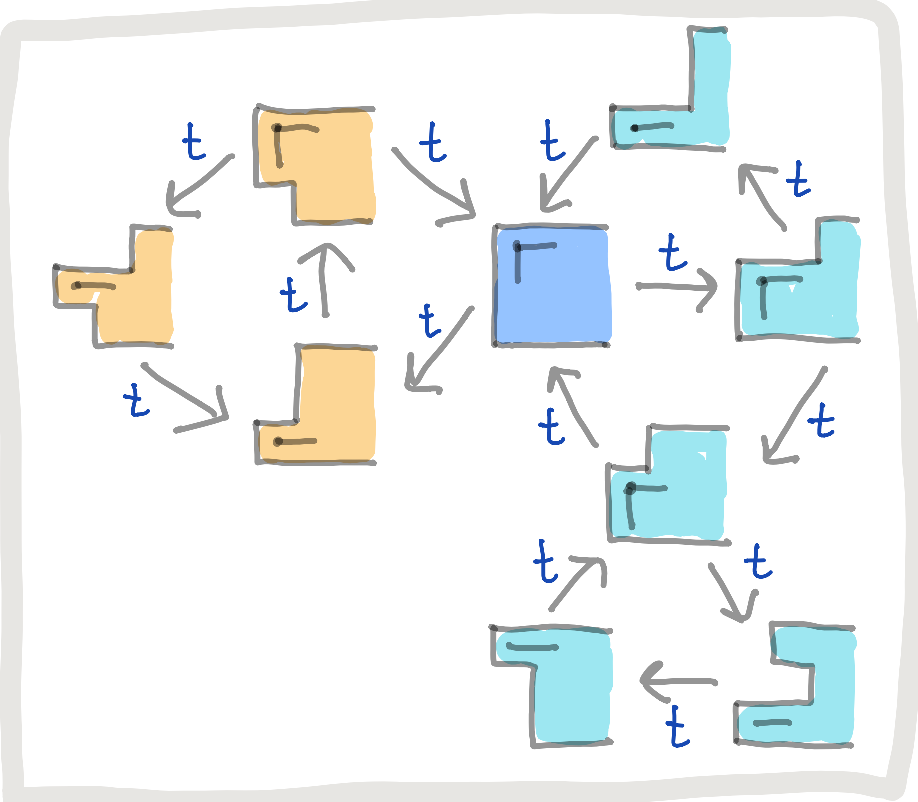
$1+t+t^2+t^3+\dots=\frac{1}{1-t}$



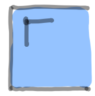
$1+(t+t^3)+(t+t^3)^2+(t+t^3)^3+(t+t^3)^4+\dots=\frac{1}{1-(t+t^3)}$

$1\quad t\ t^3\quad t^2\ 2t^4\ t^6\quad t^3\ 3t^5\ 3t^7\ t^9\quad t^4\ 4t^6\dots$

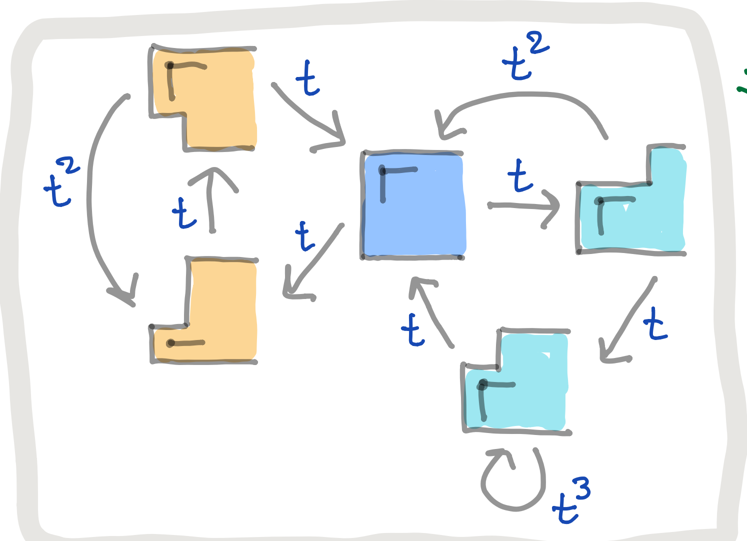
$1+t+t^2+2t^3+3t^4+\dots$

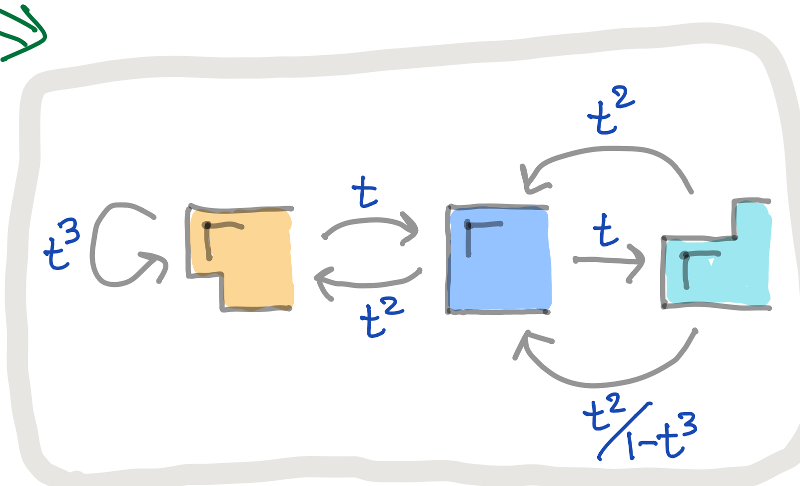


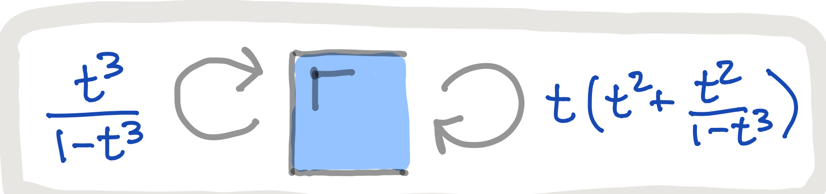
Redrawn.

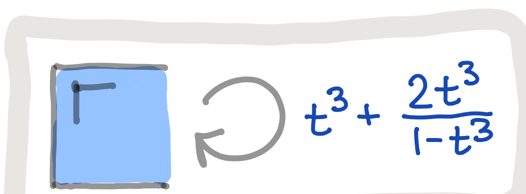
We want to sum all paths from  to itself, taking product of labels for each path.

Now simplify.







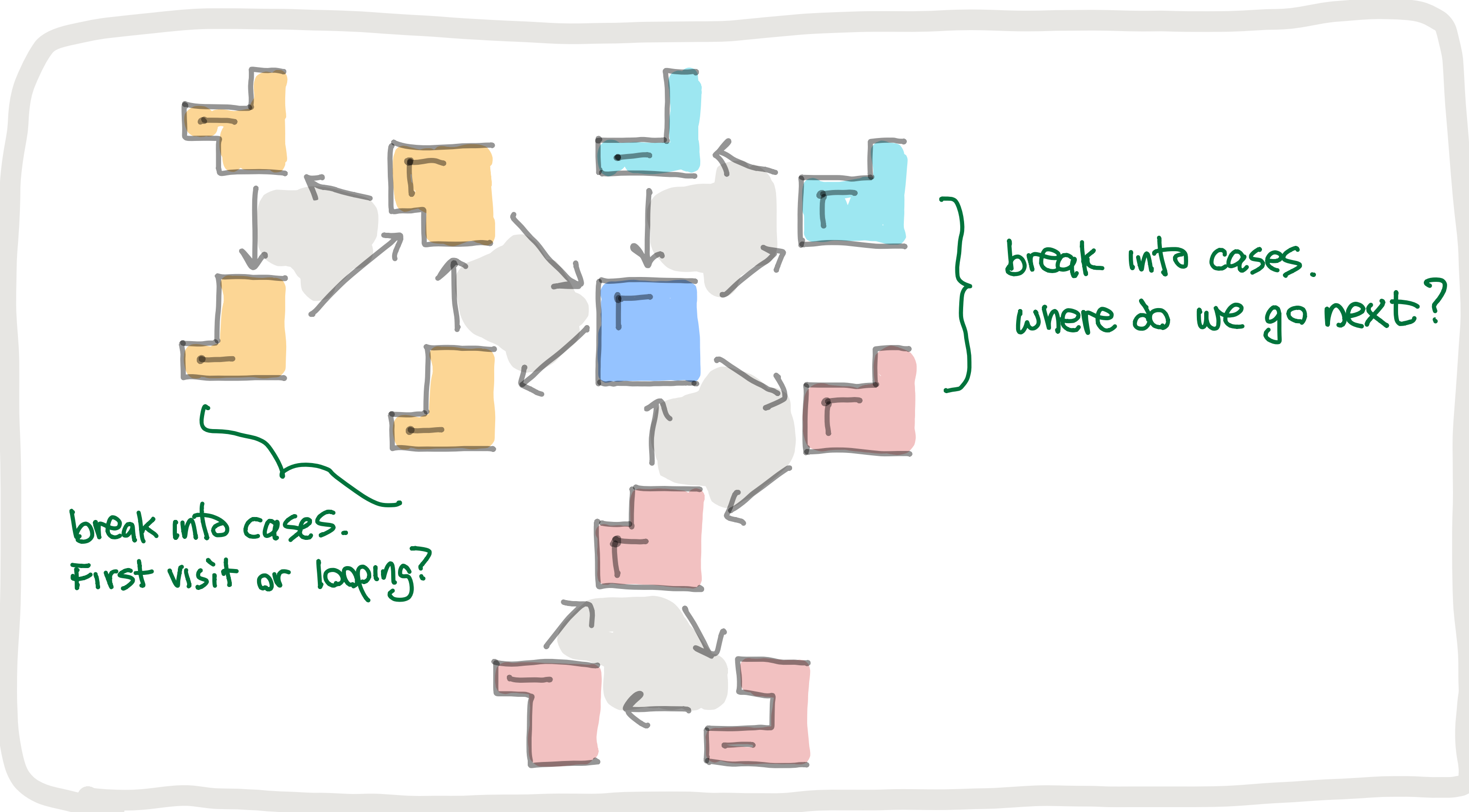


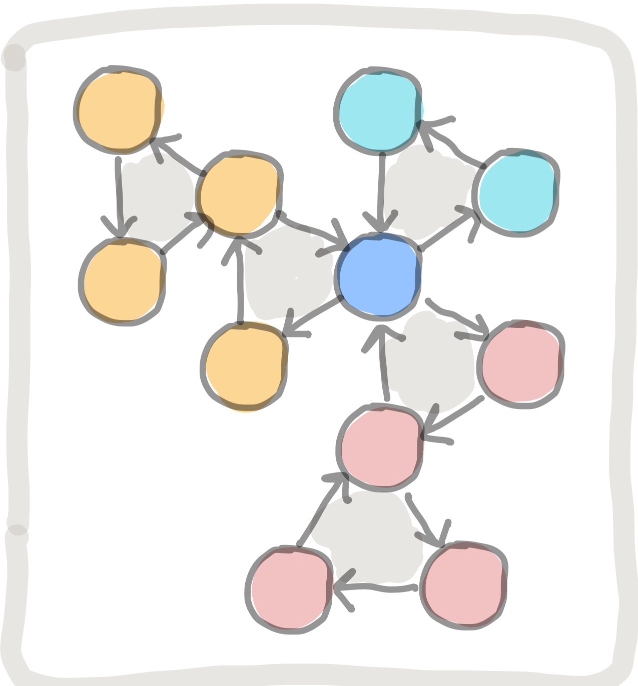
$$\boxed{1\Big/\left(1-t^3-\frac{2t^3}{1-t^3}\right)} = \frac{1-t^3}{(1-t^3)-t^3(1-t^3)-2t^3} = \frac{1-t^3}{1-4t^3+t^6}$$

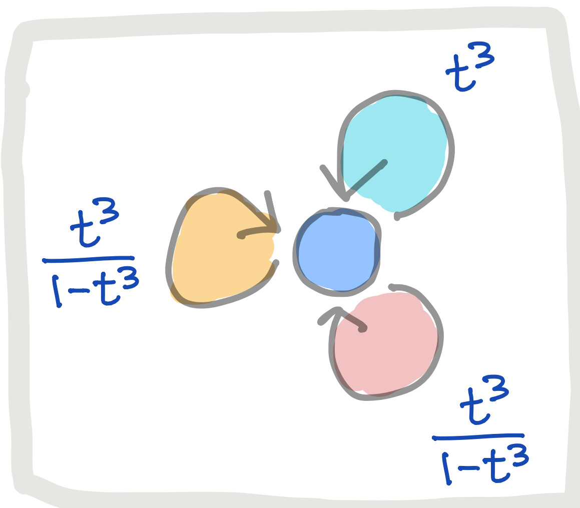
$$\boxed{\frac{1-t}{1-4t+t^2}}$$

3 dominos per n

Revised second approach: Can we make this easier to see?



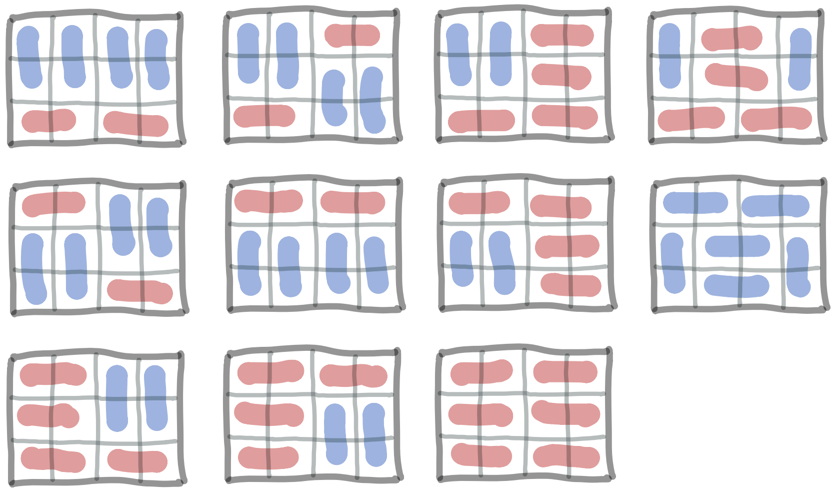


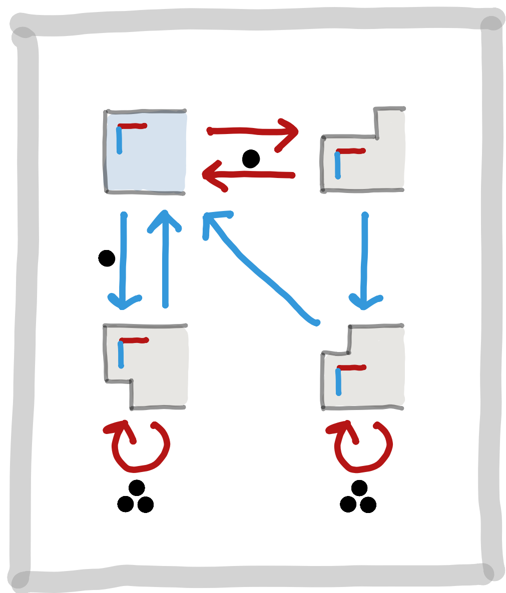


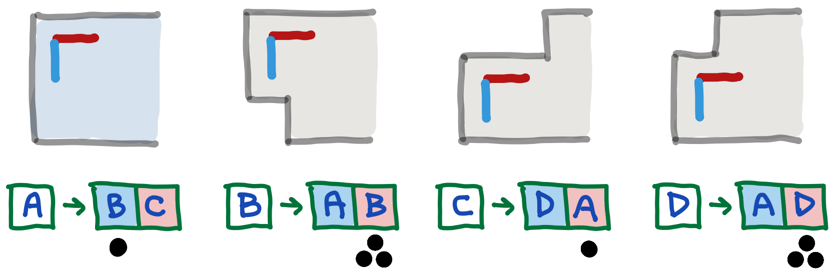
$$1\Big/\left(1-t^3-\frac{2t^3}{1-t^3}\right) = \frac{1-t}{1-4t+t^2}$$

Better to stop only when there is a choice.

Color-code edges.

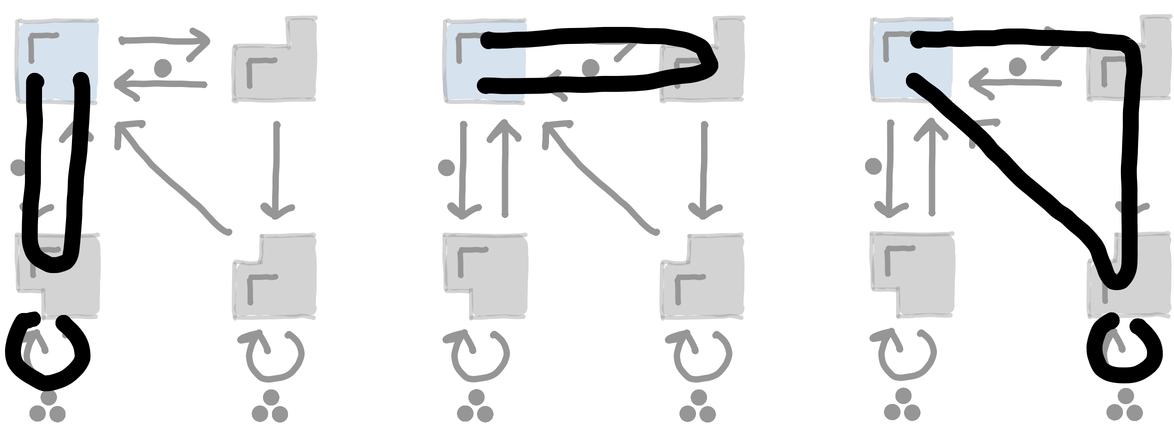






(• marks extra steps to get to a choice.)

Now classify "irreducible" walks A to A:



$t^3/1-t^3$ $\qquad t^3 \qquad$ $t^3/1-t^3$

$$\boxed{1\Big/\left(1-t^3-\frac{2t^3}{1-t^3}\right)} = \frac{1-t^3}{(1-t^3)-t^3(1-t^3)-2t^3} = \frac{1-t^3}{1-4t^3+t^6}$$

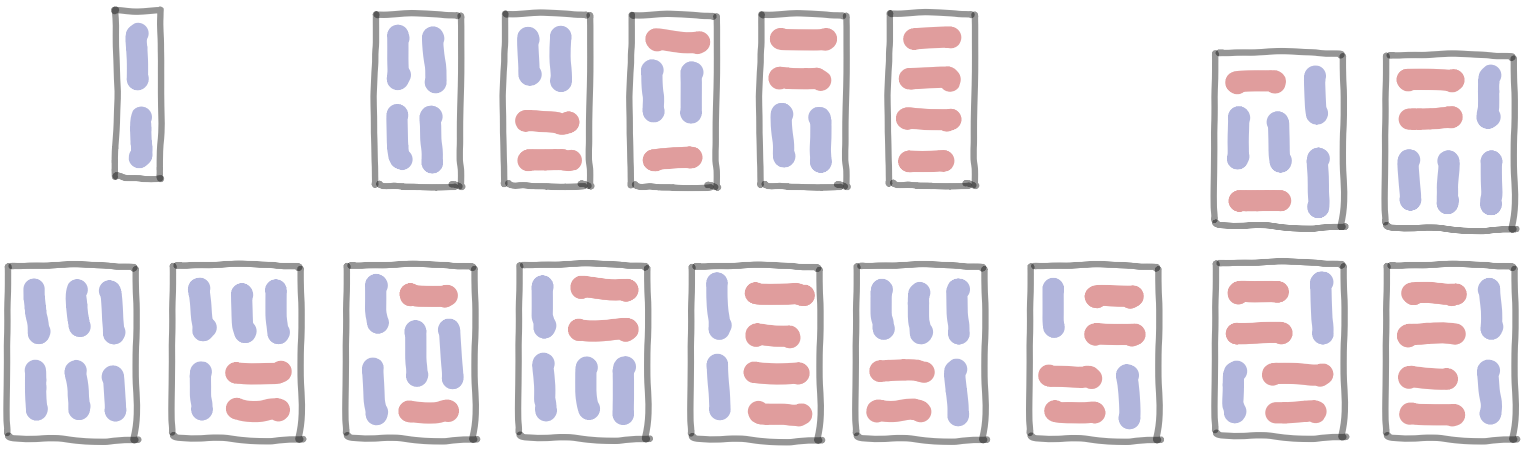
$$\boxed{\frac{1-t}{1-4t+t^2}}$$

3 dominos per n

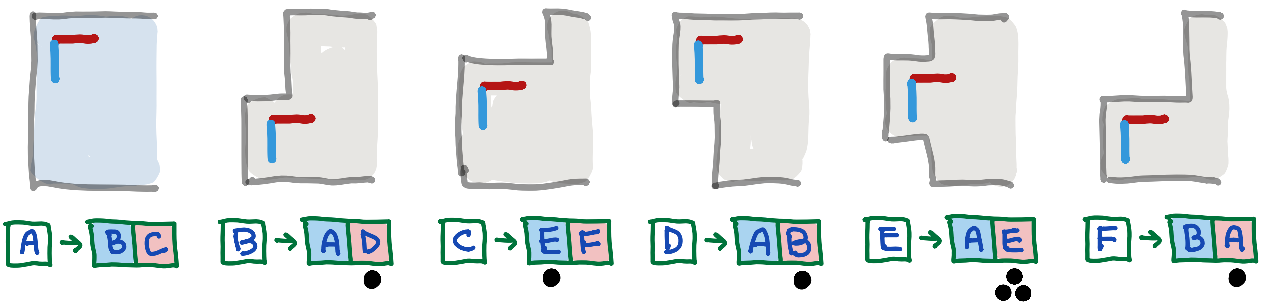
Payoff: Can we do a harder case?

Let $F(n) = \#$ of domino tilings of a $4 \times n$ grid.

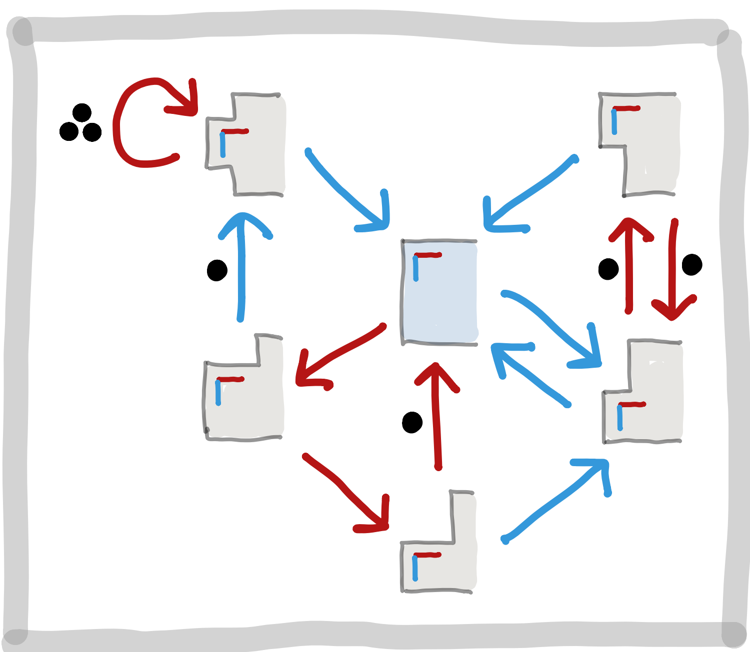
$F(1)=1$, $F(2)=5$, $F(3)=11$. Find $F(4)$, $F(5)$, generating function.



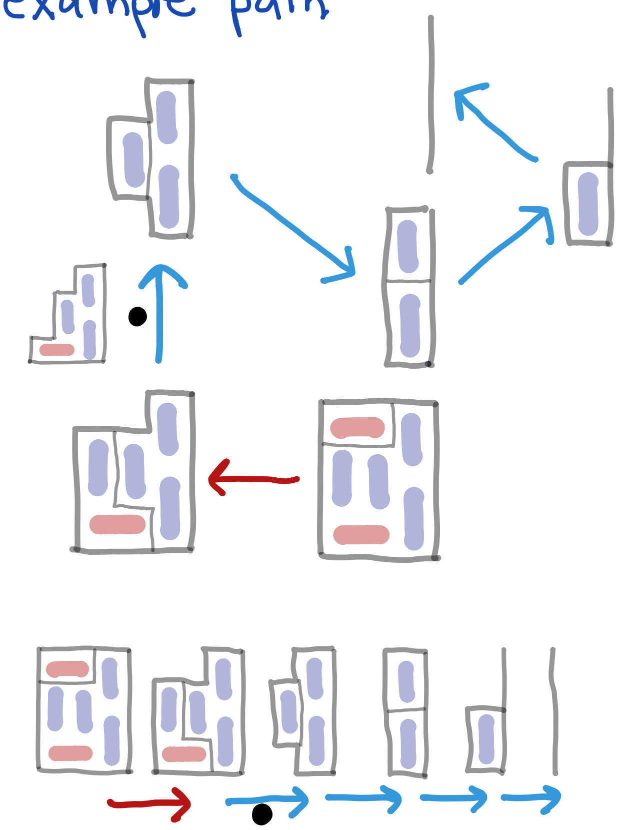
What are frontier shapes? Stop only where there's a choice.

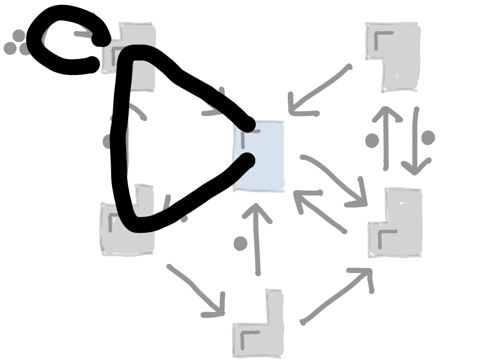


graph

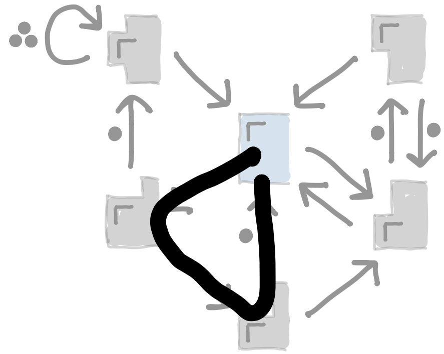


example path

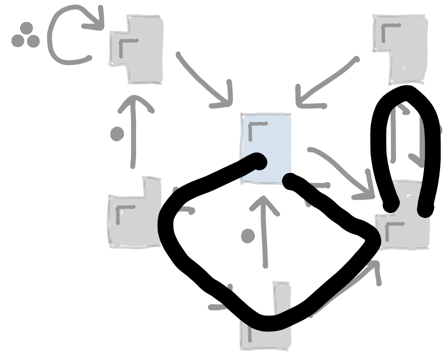




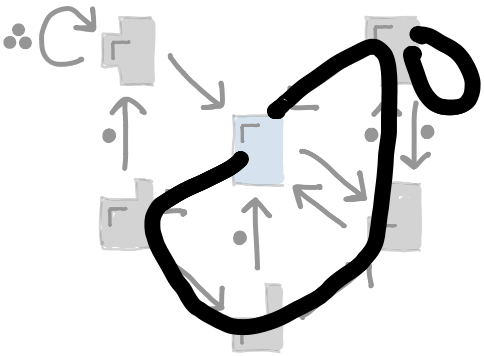
$t^4/1-t^4$



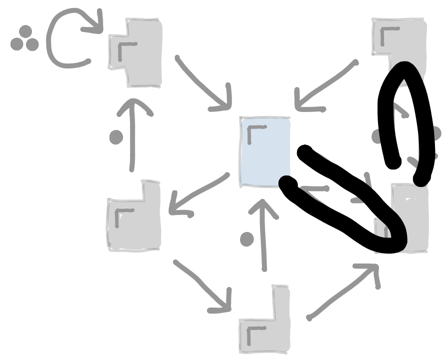
t^4



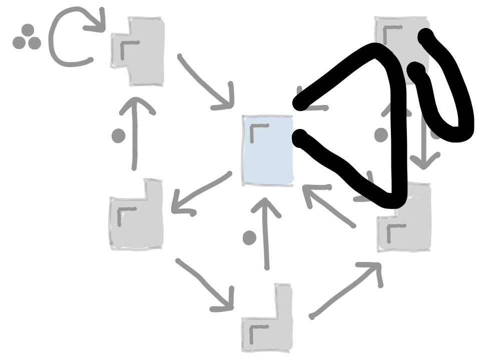
$t^4/1-t^4$



$t^6/1-t^4$



$t^2/1-t^4$



$t^4/1-t^4$

Two dominos per n, so substitute t for t^2 everywhere.

$$\boxed{\frac{1}{1-\left(t^2+\dfrac{t+3t^2+t^3}{1-t^2}\right)}}$$

$$= \boxed{\frac{1-t^2}{1-t-5t^2-t^3+t^4}}$$

dominos.nb

```
In[1]:= g = 1 / (1 - (t^2 + (t + 3 t^2 + t^3) / (1 - t^2)))
```

Out[1]= $\dfrac{1}{1-t^2-\dfrac{t+3t^2+t^3}{1-t^2}}$

```
In[2]:= g // Simplify
```

Out[2]= $\dfrac{1-t^2}{1-t-5t^2-t^3+t^4}$

```
In[3]:= Series[g, {t, 0, 5}]
```

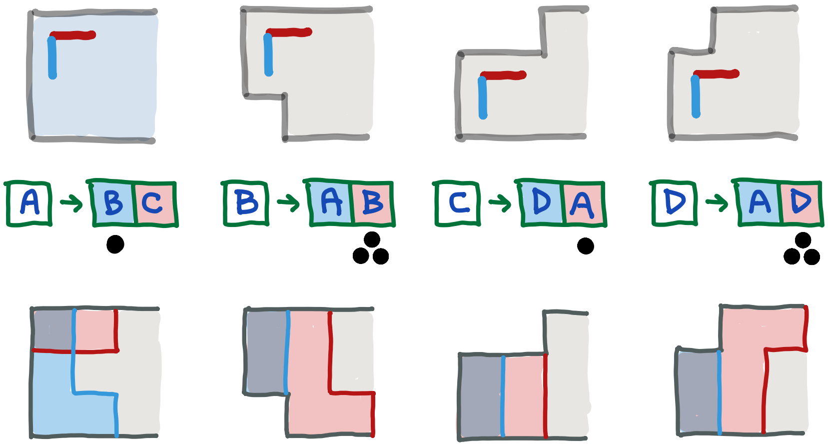
Out[3]= $1+t+5t^2+11t^3+36t^4+95t^5+O[t]^6$

or we multiply by $\dfrac{1-t^2}{1-t^2}$ $\Longleftrightarrow$ Mathematica code

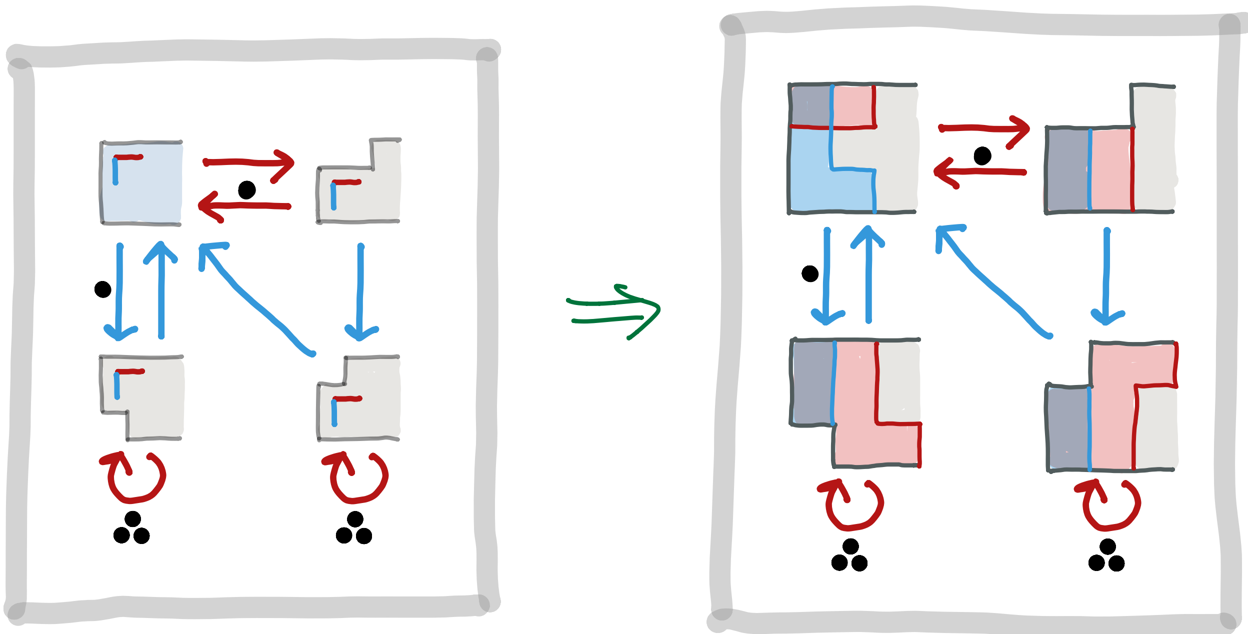
n	0	1	2	3	4	5	...
$1/(1-t-5t^2-t^3+t^4)$	1	1	6	12	42	107	...
$-t^2/(1-t-5t^2-t^3+t^4)$	0	0	1	1	6	12	...
$f(n)$	1	1	5	11	36	95	...

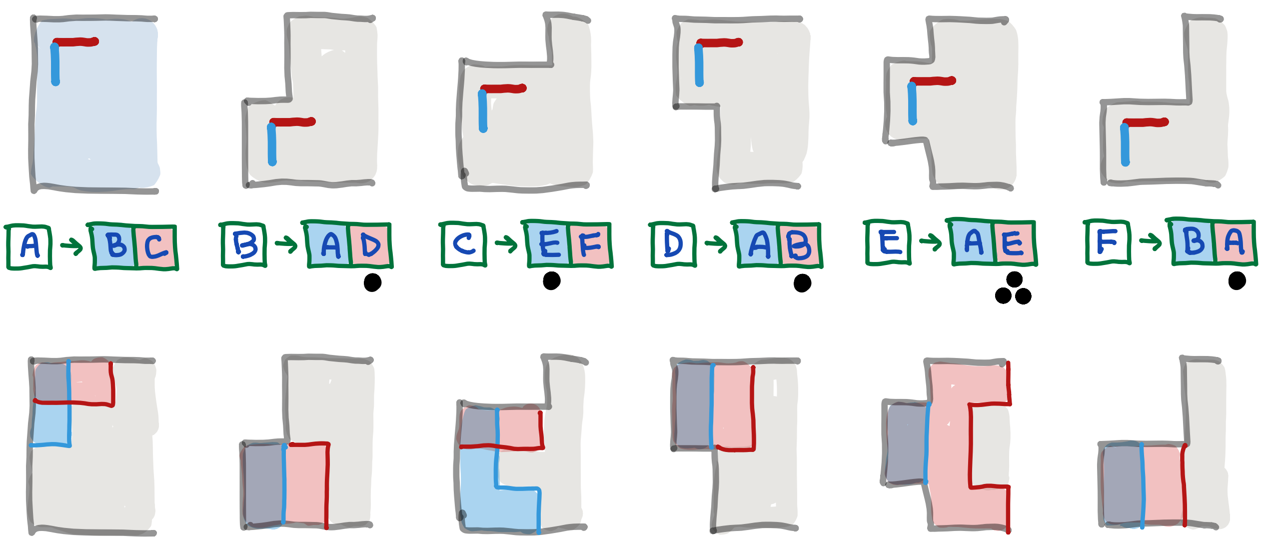
use recurrence
$1=t+5t^2+t^3-t^4$
shift by t^2

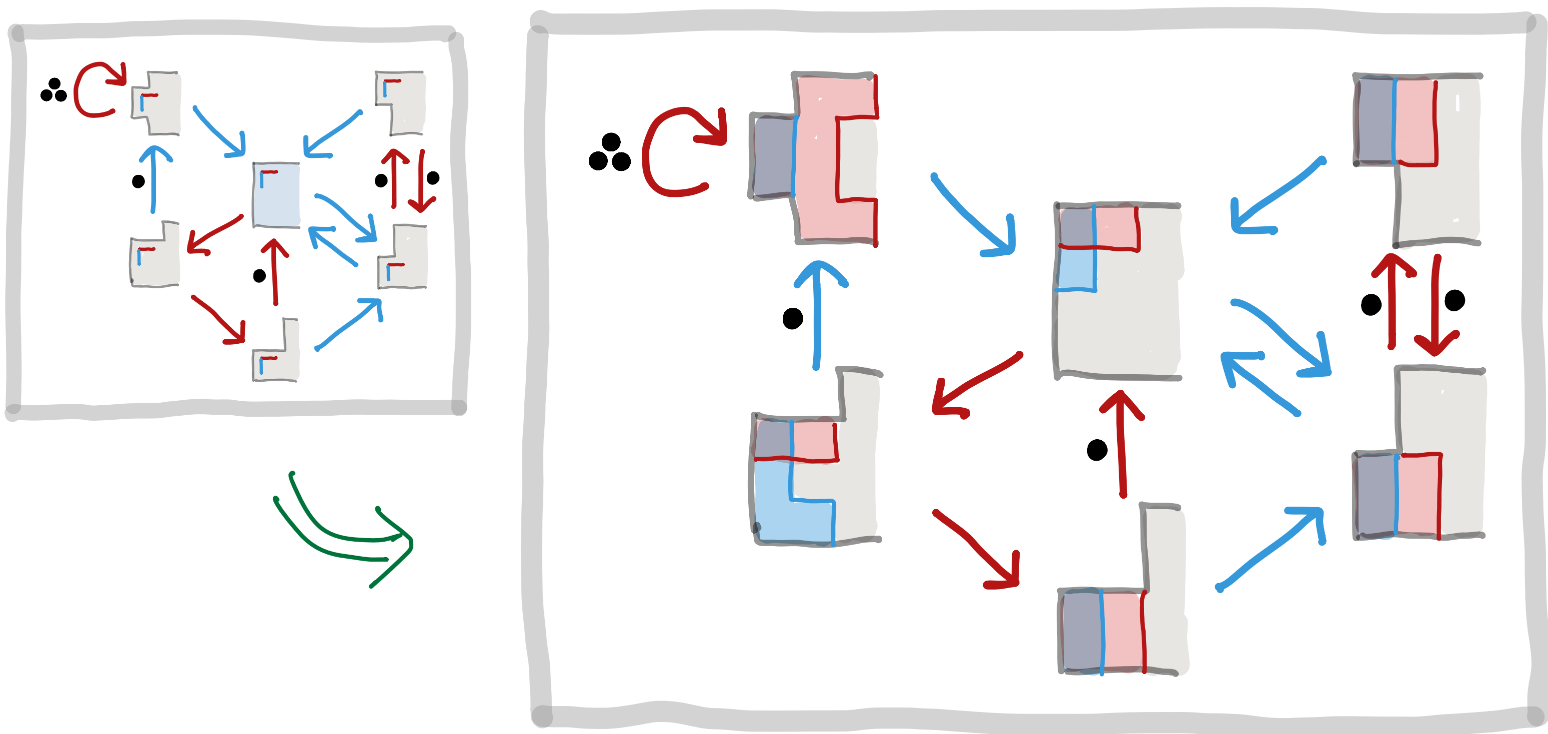
$\boxed{\text{OEIS A005178}}$

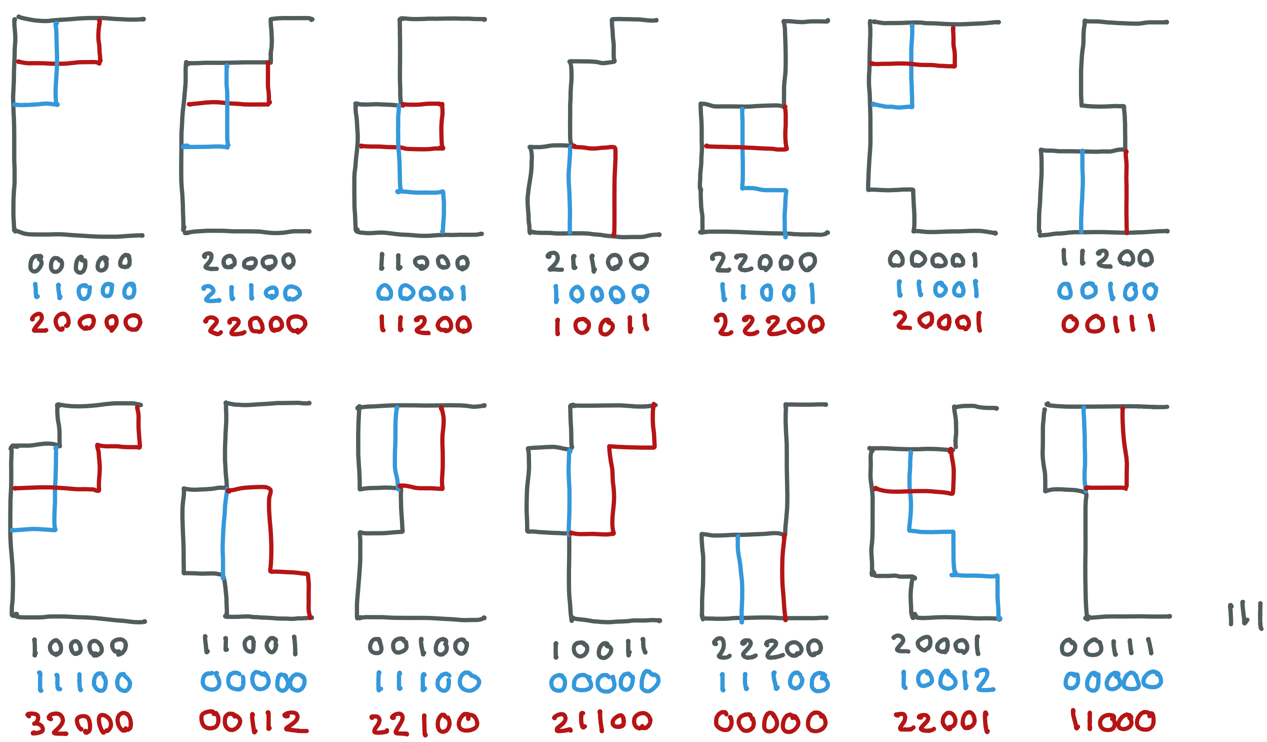


Processing these tables is a strain. Can we make this easier to see?







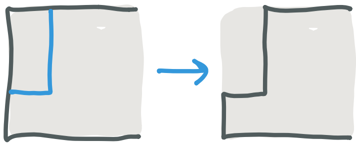


Easier to learn rules for 00000 representation without diagrams

Need to track # of moves for ∴ labels

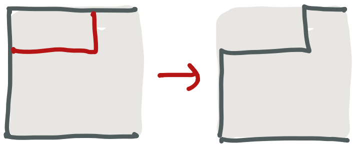
(Of course we could also switch to a computer...)

Relearn 4xn case in preparation for 5xn case.



0 0 0
1 1 0

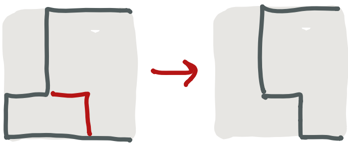
Increment first pair of zeros



0 0 0
2 0 0

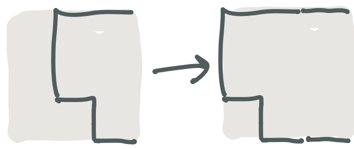
Add 2 to first zero

What are rules?



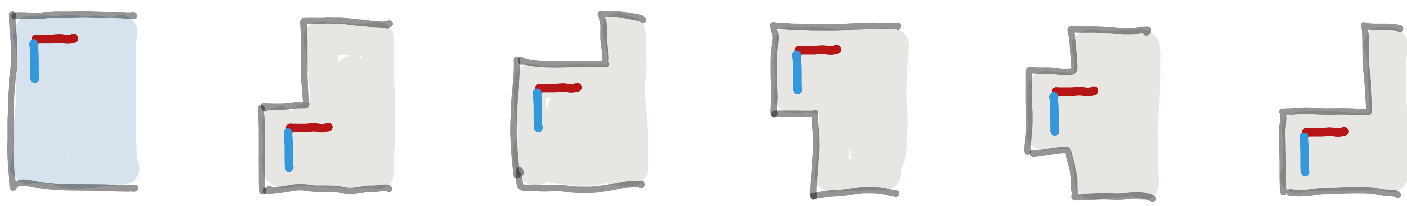
1 1 0
1 1 2 •

Isolated zero is forced move



1 1 2
0 0 1

Decrement if no zeros



A → B C

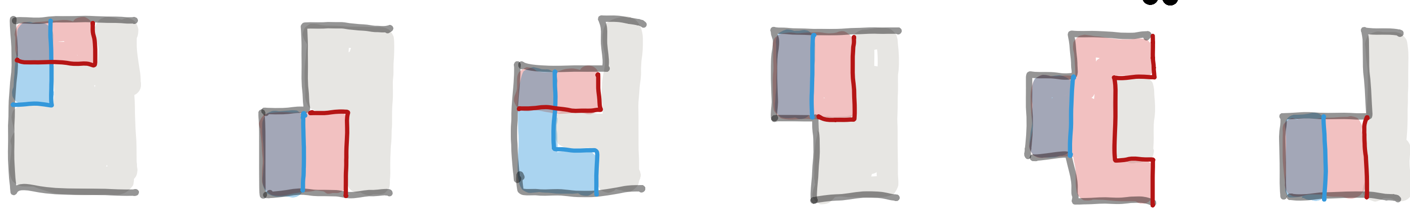
B → A D •

C → E F •

D → A B •

E → A E ∴

F → B A •



0 0 0 0
1 1 0 0
2 0 0 0

1 1 0 0
0 0 0 0
0 0 1 1 2

2 0 0 0
1 0 0 1 2
2 2 0 0

0 0 1 1
0 0 0 0
1 1 0 0 2

1 0 0 1
0 0 0 0
1 0 0 1 4

2 2 0 0
1 1 0 0
0 0 0 0 2

1 1 0 0
1 1 2 0
1 1 2 2 •
0 0 1 1

2 0 0 0
2 1 1 0
2 1 1 2 •
1 0 0 1

0 0 1 1
2 0 1 1
2 2 1 1 •
1 1 0 0

1 0 0 1
1 2 0 1
1 2 2 1 •
0 1 1 0
2 1 1 0 •
2 1 1 2 •
1 0 0 1

2 2 0 0
2 2 2 0
2 2 2 2 •
1 1 1 1
0 0 0 0

Now, 5xn case (confirms we've reached that table.)

0	0	0	0	0	
1	1	0	0	0	
2	0	0	0	0	

0	0	0	0	1	
1	1	0	0	1	
2	0	0	0	1	

0	0	0	1	1	
0	0	1	0	0	2
2	0	0	1	1	

0	0	1	0	0	
1	1	1	0	0	
2	2	1	0	0	2

0	0	1	1	1	
0	0	0	0	0	
1	1	0	0	0	2

0	0	1	1	2	
0	0	0	0	1	
1	1	0	0	1	2

1	0	0	0	0	
1	1	1	0	0	
1	2	0	0	0	

1	0	0	1	1	
0	0	0	0	0	
2	1	1	0	0	3

1	0	0	1	2	
0	0	0	0	1	
1	0	0	1	2	5

1	1	0	0	0	
0	0	0	0	1	2
1	1	2	0	0	

1	1	0	0	1	
0	0	0	0	0	
0	0	1	1	2	3

1	1	1	0	0	
0	0	0	0	0	
0	0	0	1	1	2

1	1	2	0	0	
0	0	1	0	0	
0	0	1	1	1	2

1	2	0	0	0	
2	1	0	0	1	3
1	2	2	0	0	

1	2	2	0	0	
2	1	1	0	0	2
1	0	0	0	0	3

2	0	0	0	0	
2	1	1	0	0	
2	2	0	0	0	

2	0	0	0	1	
1	0	0	1	2	3
2	2	0	0	1	

2	0	0	1	1	
1	0	0	0	0	
1	1	1	0	0	2

2	1	0	0	1	
1	0	0	0	0	
2	1	0	0	1	5

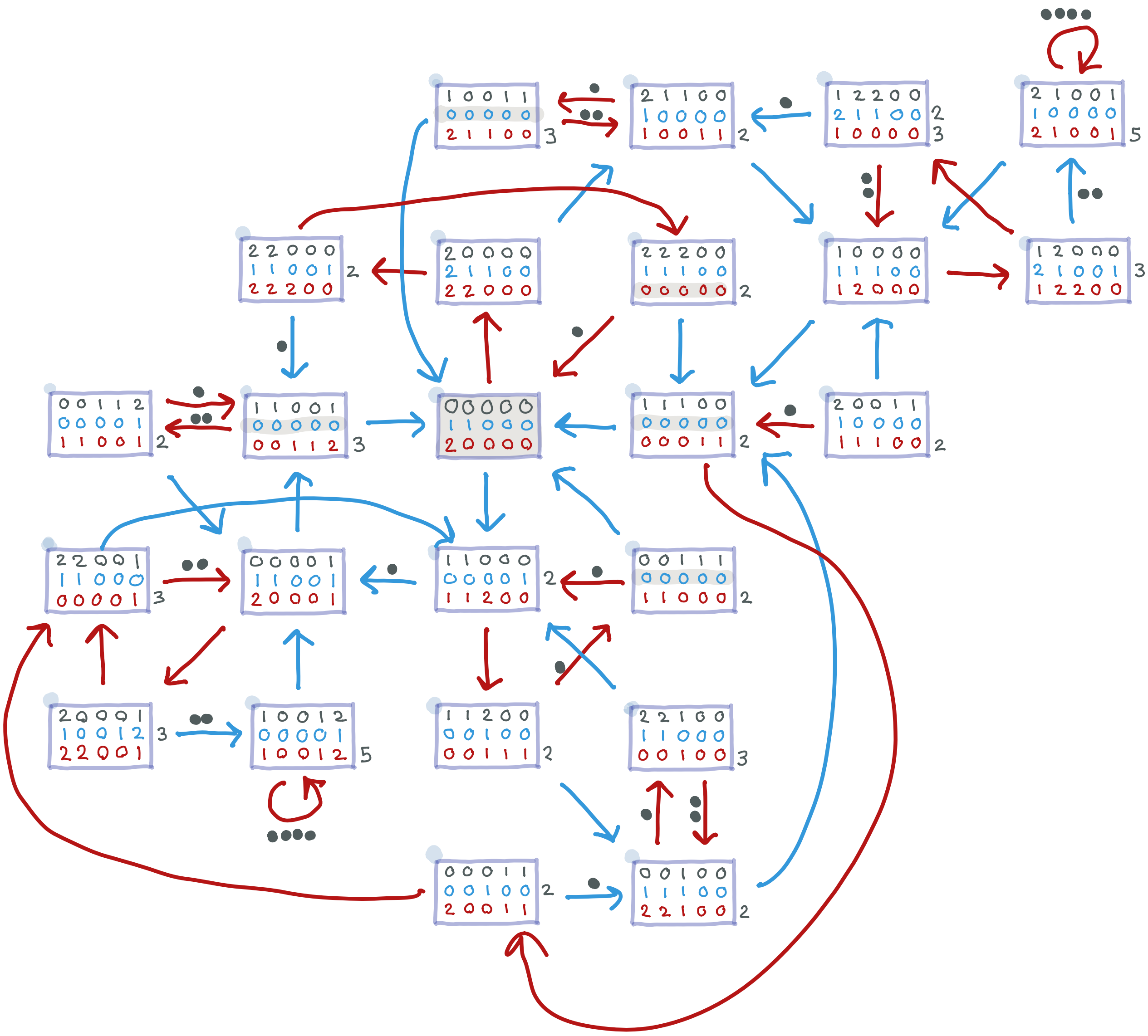
2	1	1	0	0	
1	0	0	0	0	
1	0	0	1	1	2

2	2	0	0	0	
1	1	0	0	1	2
2	2	2	0	0	

2	2	0	0	1	
1	1	0	0	0	
0	0	0	0	1	3

2	2	1	0	0	
1	1	0	0	0	
0	0	1	0	0	3

2	2	2	0	0	
1	1	1	0	0	
0	0	0	0	0	2



We're well past the point where we should have switched to a computer.

At least the process is in a form that's easy to program.

This wasn't easy, but the fact it is possible shows us this kind of counting problem can be mechanized.

Generalization: **Finite State Automata**