Aigner, p242 Burnside's lemma

Counting with symmetry

$$\sum_{x \in X} |G_x| = \sum_{g \in G} |X_g|\,. \tag{1}$$

Lemma 6.1. *Let G act on X. Then for any $x \in X$,*

$$|M(x)| = \frac{|G|}{|G_x|}\,. \tag{2}$$

Lemma 6.2 (Burnside–Frobenius). *Let the group G act on X, and let $\mathcal{M}$ be the set of patterns. Then*

$$|\mathcal{M}| = \frac{1}{|G|} \sum_{g \in G} |X_g|\,. \tag{3}$$

We need to understand how to read this.

X = raw set of objects

G = symmetries acting on X

M = patterns, equivalence classes of objects up to symmetry

X_g = elements of X fixed by $g \in G$

Example: X = length 2 lists from $\{a,b\}$ $\qquad G = \{ \boxed{1}, \boxed{\leftrightarrow} \}$

1: do nothing $\qquad \leftrightarrow$: flip

$X = \{ \boxed{a,a} \ \boxed{a,b} \ \boxed{b,a} \ \boxed{b,b} \}$

$M = \{ \boxed{\boxed{a,a}} \ \boxed{\boxed{a,b} \ \boxed{b,a}} \ \boxed{\boxed{b,b}} \}$

$|X| = 4$

$|G| = 2$

$|M| = 3$

$\boxed{a,a} \qquad \boxed{a,b \quad b,a} \qquad \boxed{b,b}$

M = "orbits" of action of G on X

$X_1 = \{ \boxed{a,a} \ \boxed{a,b} \ \boxed{b,a} \ \boxed{b,b} \}$ $\qquad |X_1| = 4$

$X_{\leftrightarrow} = \{ \boxed{a,a} \ \boxed{b,b} \}$ $\qquad |X_{\leftrightarrow}| = 2$

$$\boxed{\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}\left(|X_1| + |X_{\leftrightarrow}|\right) = \frac{1}{2}(4+2) = 3 = |M|}$$

Example: X = length 3 lists from $\{a,b,c\}$

$G = \{ \boxed{1}, \boxed{\leftrightarrow} \}$

do nothing $\quad$ flip

$|X| = 27 \qquad |X_1| = 27$

$|G| = 2 \qquad |X_{\leftrightarrow}| = 9$

$|M| = 18$

$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{2}\left(|X_1| + |X_{\leftrightarrow}|\right) = \frac{1}{2}(27+9) = 18 = |M|}$$

Example: $X =$ length k lists from $\{a_1, \dots, a_n\}$

$G = \{ \boxed{1}, \boxed{\leftrightarrow} \}$

do nothing   flip $\rightarrow$

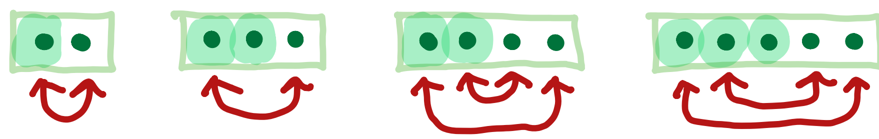
$|X| = n^k = |X_1|$

$|G| = 2$

$|X_{\leftrightarrow}| = n^{\lceil \frac{k}{2} \rceil}$

substep: do a counting problem

$|X_{\leftrightarrow}| = n^{\lceil \frac{k}{2} \rceil}$ $(\lceil \frac{k}{2} \rceil =$ round up $k/2)$



1 choice	2 choices	2 choices	3 choices
n	n^2	n^2	n^3

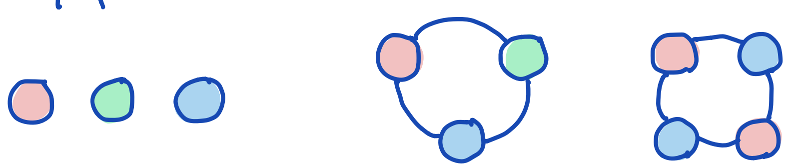
$$\boxed{\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}(|X_1| + |X_{\leftrightarrow}|) = \frac{1}{2}(n^k + n^{\lceil \frac{k}{2} \rceil}) = |M|}$$

$n = k = 2$ $\frac{1}{2}(2^2 + 2) = 3$ ☑

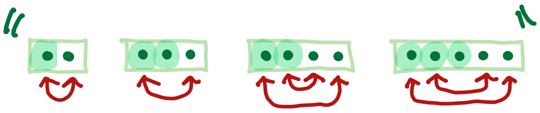
$n = k = 3$ $\frac{1}{2}(3^3 + 3^2) = 18$ ☑

Example: "Necklace" problems

Make an n-bead necklace using k possible colors of beads. Two patterns are the same if they agree after rotation. How many patterns?



For each n, there will be a version of the "



"

$n^{\lceil \frac{k}{2} \rceil}$ counting problem

Divisibility = more symmetry

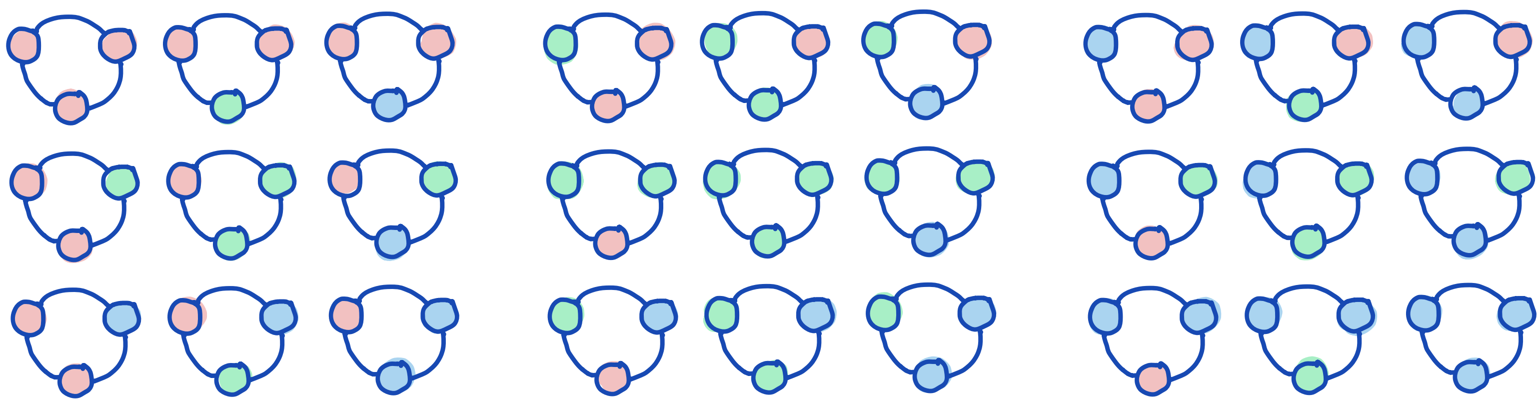
$n = k = 3$

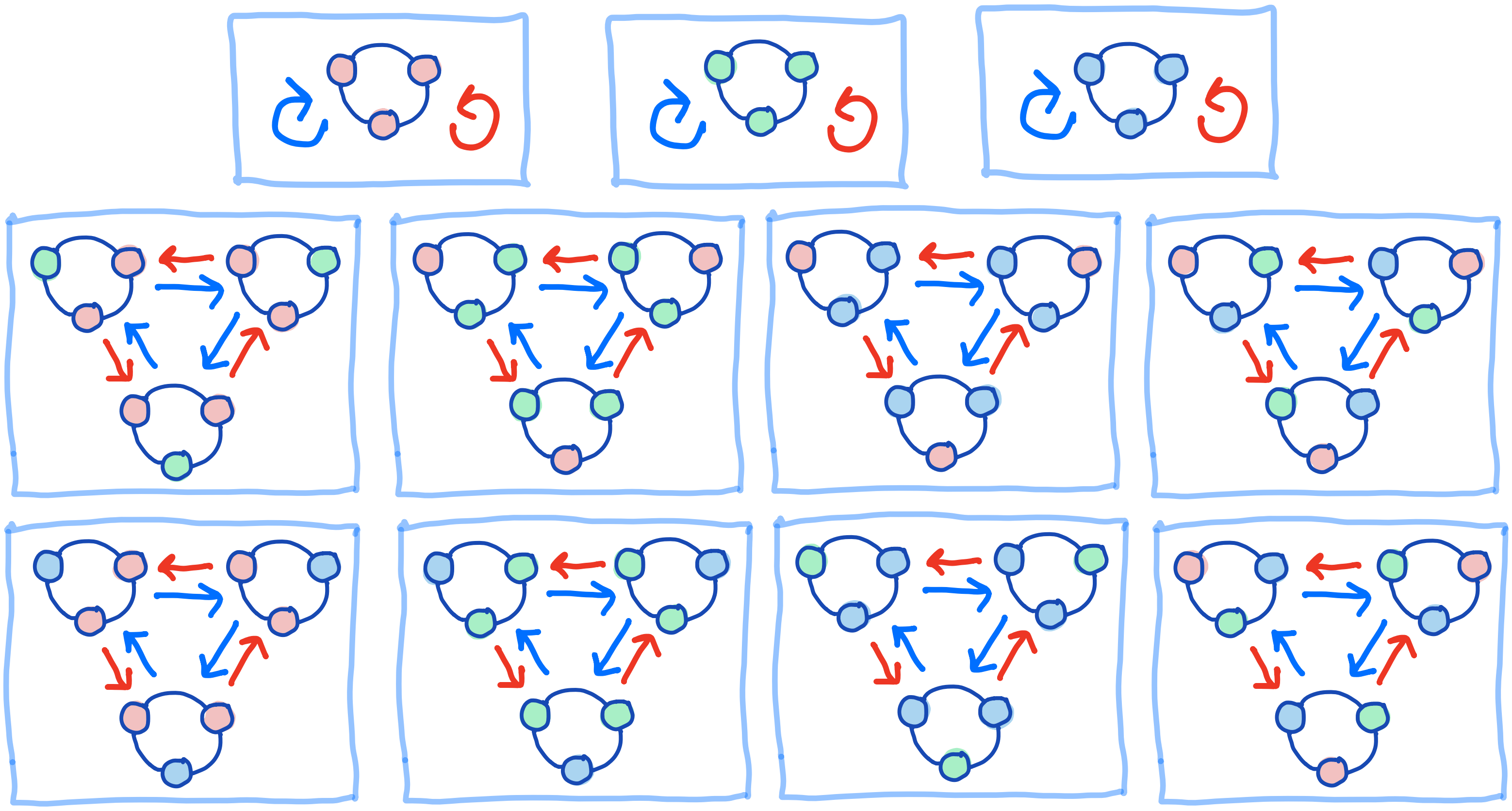
$G = \{ \boxed{1}, \boxed{\circlearrowright}, \boxed{\circlearrowleft} \}$

do nothing

1/3 turn →

1/3 turn →





$|G| = 3 \qquad |X| = 27 = |X_1| \qquad |X_{\circlearrowright}| = |X_{\circlearrowleft}| = 3$

$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{3}\left(|X_1| + |X_{\circlearrowright}| + |X_{\circlearrowleft}|\right) = \frac{1}{3}(27+3+3) = 11}$$

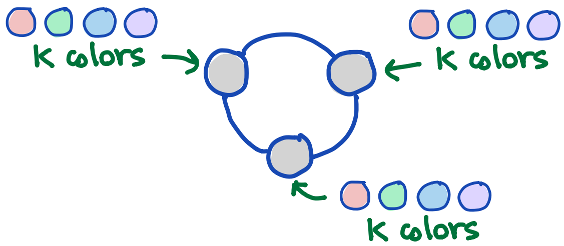
☑

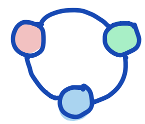
$n = 3$ any k

$G = \{\, \boxed{1}, \boxed{\circlearrowright}, \boxed{\circlearrowleft} \,\}$

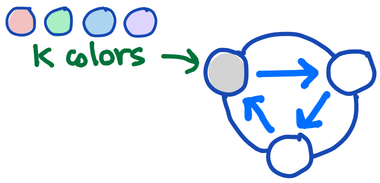
do nothing · $\frac{1}{3}$ turn · $\frac{1}{3}$ turn

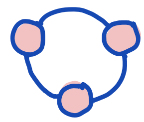
$$|X| = |X_1| = k^3$$



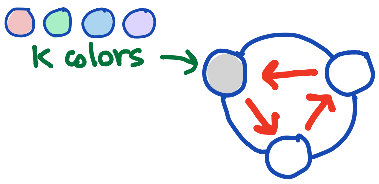


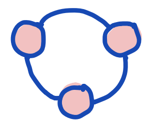
$$|X_{\circlearrowright}| = k$$





$$|X_{\circlearrowleft}| = k$$

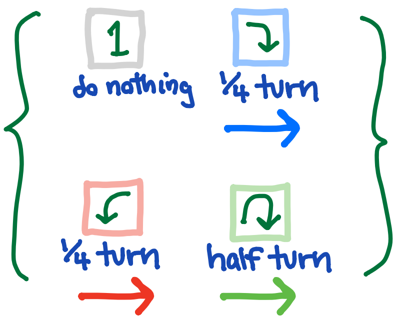


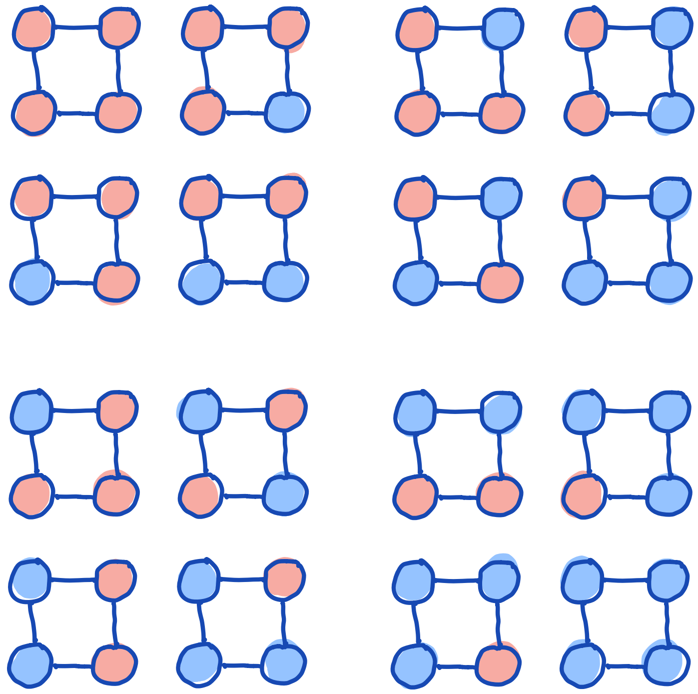


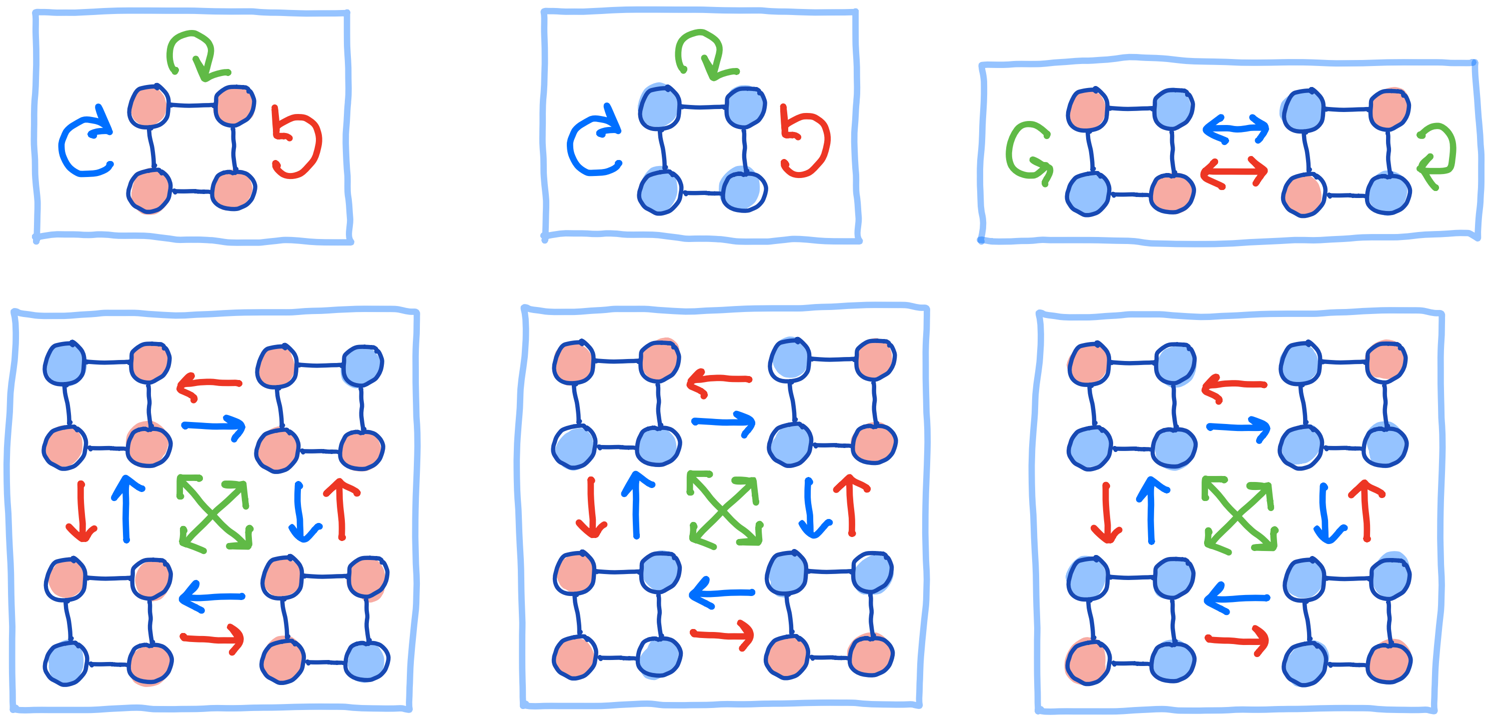
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{3}\left(|X_1| + |X_{\circlearrowright}| + |X_{\circlearrowleft}|\right) = \frac{1}{3}\left(k^3 + k + k\right)}$$

Check: $k = 3$ $\quad \frac{1}{3}\left(k^3 + k + k\right) = \frac{1}{3}(27 + 3 + 3) = 11$ ☑

$n=4 \quad k=2$

$G =$ 



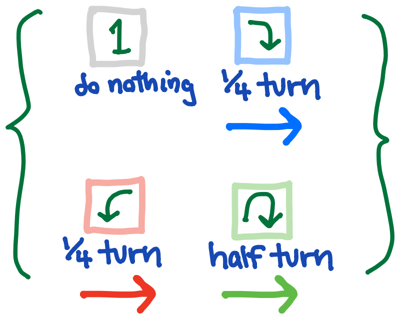


$|G| = 4 \qquad |X| = 16 = |X_1| \qquad |X_{\curvearrowright}| = |X_{\curvearrowleft}| = 2 \qquad |X_{\circlearrowright}| = 4$

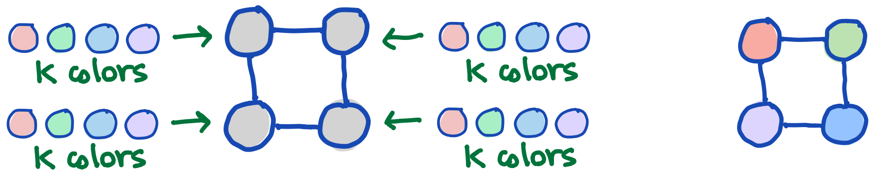
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{4}\left(|X_1|+|X_{\curvearrowright}|+|X_{\curvearrowleft}|+|X_{\circlearrowright}|\right) = \frac{1}{4}(16+2+2+4) = 6}$$ ☑

$n = 4$ any k

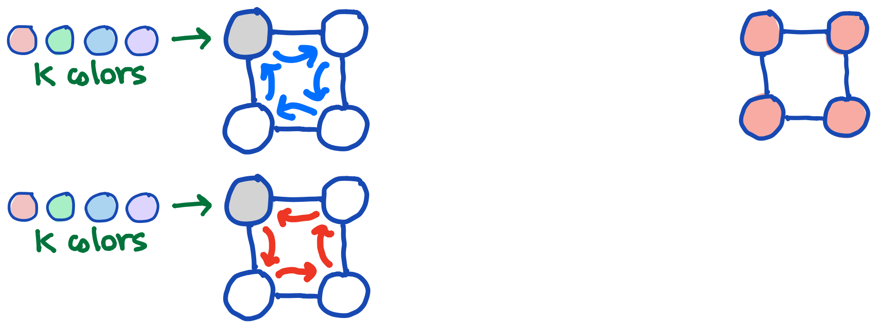
$G =$



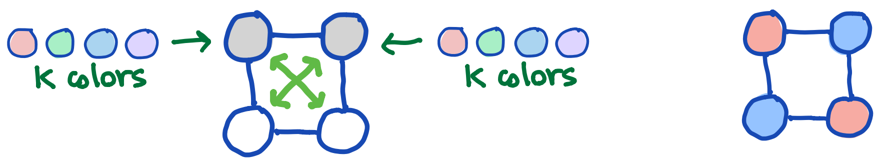
$|X| = |X_1| = k^4$



$|X_{\curvearrowright}| = |X_{\curvearrowleft}| = k$



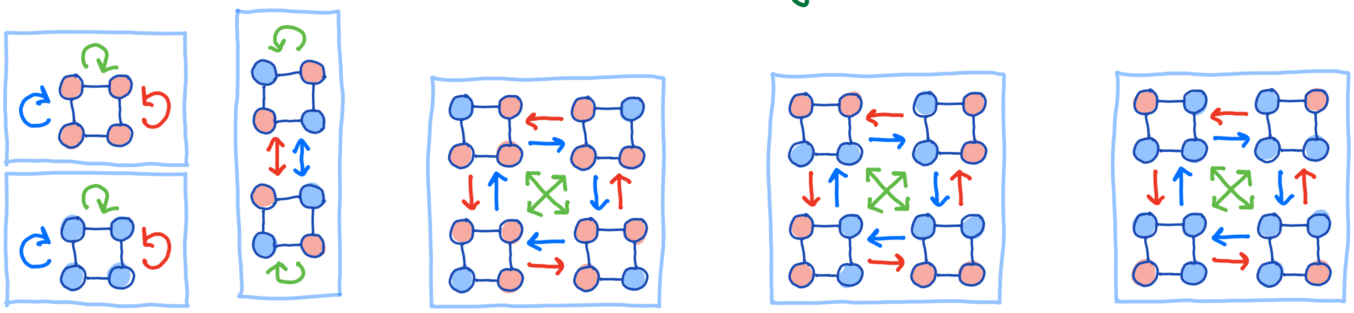
$|X_{\circlearrowright}| = k^2$

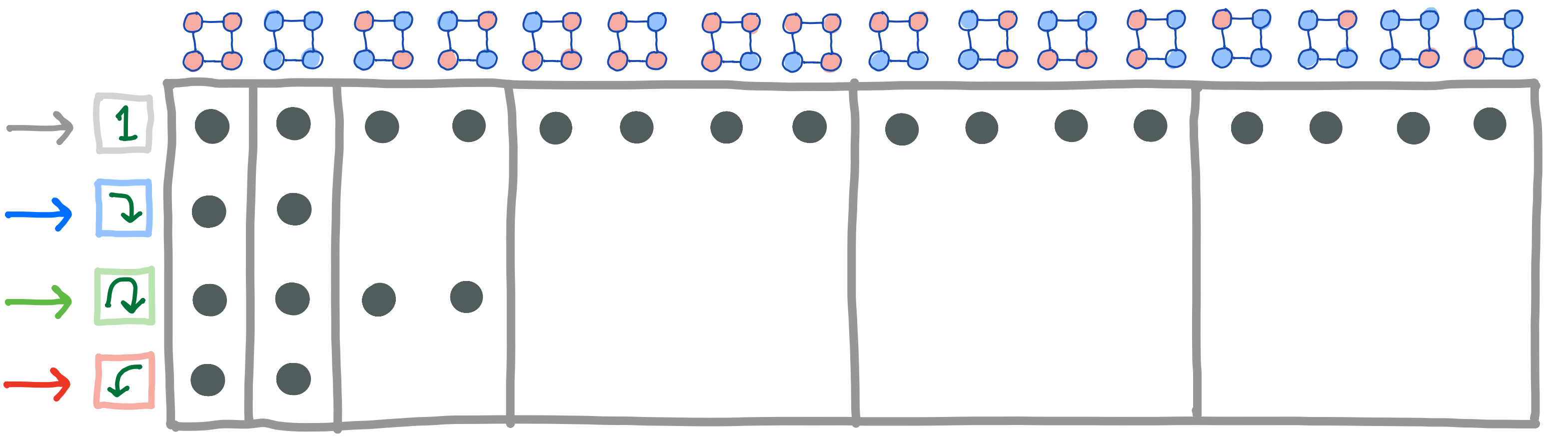


$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{4}\left(|X_1|+|X_{\curvearrowright}|+|X_{\curvearrowleft}|+|X_{\circlearrowright}|\right) = \frac{1}{4}\left(k^4+k+k+k^2\right)}$$

Check: $k = 2$ $\quad \frac{1}{4}\left(k^4+k+k+k^2\right) = \frac{1}{4}(16+2+2+4) = 6$ ☑

Why does this work? $\qquad \frac{1}{|G|}\sum_{g\in G}|X_g| = |M|$

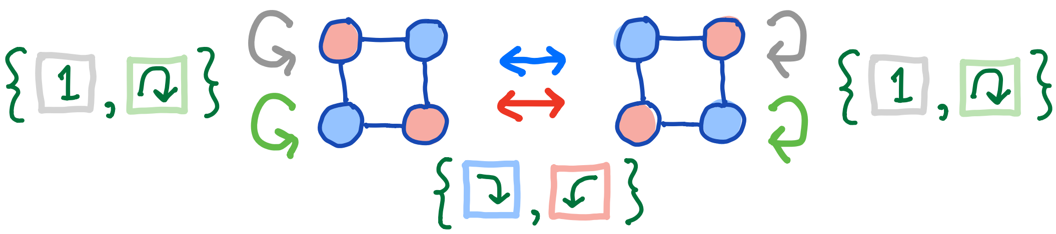




Each dot ● marks an object fixed b a group element.
Each box is a pattern up to symmetry.
The row sums are $|X_1|, |X_{\text{↴}}|, |X_{\text{↷}}|, |X_{\text{↶}}|$.
If we can figure out why each box gets $|G|$ dots, we're done.

Group Theory in a nutshell: things divide up evenly.

Look more closely at each orbit. This one is interesting:



$G_{\square} = \{\boxed{1}, \boxed{\text{↷}}\}$ = elements of G that fix $\square$

$\boxed{\text{↴}}\,G_{\square} = \boxed{\text{↴}}\{\boxed{1}, \boxed{\text{↷}}\} = \{\underbrace{\boxed{\text{↴}}\boxed{1}}_{\boxed{\text{↴}}}, \underbrace{\boxed{\text{↴}}\boxed{\text{↷}}}_{\boxed{\text{↶}}}\} = \{\boxed{\text{↴}}, \boxed{\text{↶}}\}$

$$\boxed{|\{\boxed{1}, \boxed{\text{↷}}\}|\,|\{\square, \square\}| = |\{\boxed{1}, \boxed{\text{↴}}, \boxed{\text{↷}}, \boxed{\text{↶}}\}| = |G|}$$