Combinatorics Feb23

What is a group?

One operation $*$ or $+$
Identity and inverses
Associative: $(ab)c = a(bc)$

$\mathbb{Z}_2$:

+	0	1
0	0	1
1	1	0

mod 2

≈

+	even	odd
even	even	odd
odd	odd	even

≈

*	1	-1
1	1	-1
-1	-1	1

≈

*	1	2
1	1	2
2	2	1

mod 3

$\mathbb{Z}_3$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

mod 3

$\mathbb{Z}_4$:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

mod 4

≈

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

mod 5

+		*
0	↔	1
1	↔	2
2		3
3		4

$\mathbb{Z}_2 \times \mathbb{Z}_2$:

+	0,0	0,1	1,0	1,1
0,0	0,0	0,1	1,0	1,1
0,1	0,1	0,0	1,1	1,0
1,0	1,0	1,1	0,0	0,1
1,1	1,1	1,0	0,1	0,0

mod 2,2

$\mathbb{Z}_5$:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\mathbb{Z}_6$:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

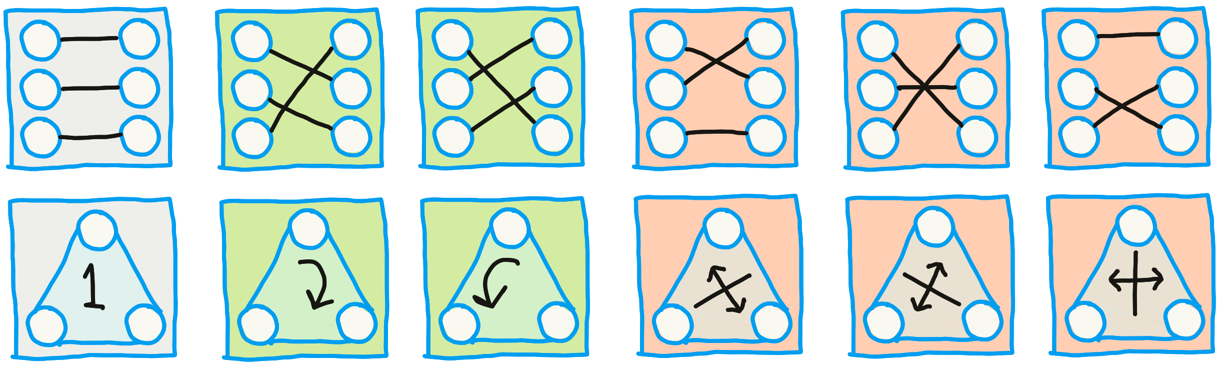
$\mathbb{Z}_2 \times \mathbb{Z}_3$:

+	0,0	0,1	0,2	1,0	1,1	1,2
0,0	0,0	0,1	0,2	1,0	1,1	1,2
0,1	0,1	0,2	0,0	1,1	1,2	1,0
0,2	0,2	0,0	0,1	1,2	1,0	1,1
1,0	1,0	1,1	1,2	0,0	0,1	0,2
1,1	1,1	1,2	1,0	0,1	0,2	0,0
1,2	1,2	1,0	1,1	0,2	0,0	0,1

mod 2,3

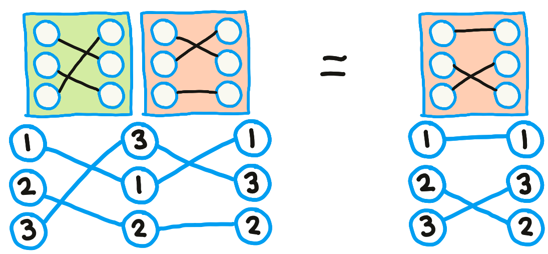
Inverses $\Leftrightarrow$ Each row is a permutation of the first row
Each col is a permutation of the first col

The symmetric group S_3: Permutations of $\{1,2,3\}$
Symmetries of a triangle

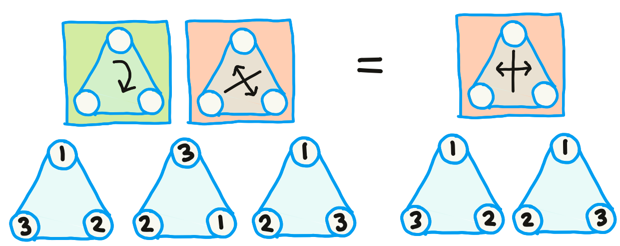


How to multiply?

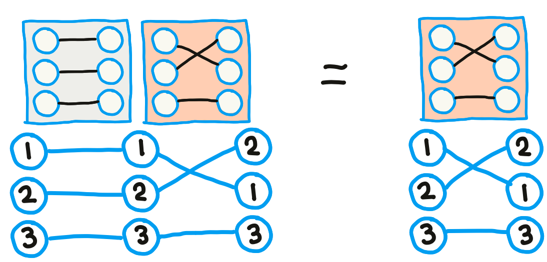
Pull tight



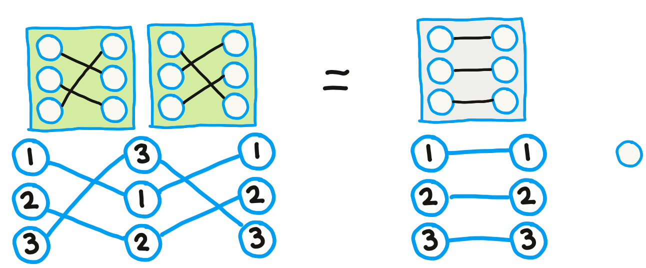
Watch test triangle



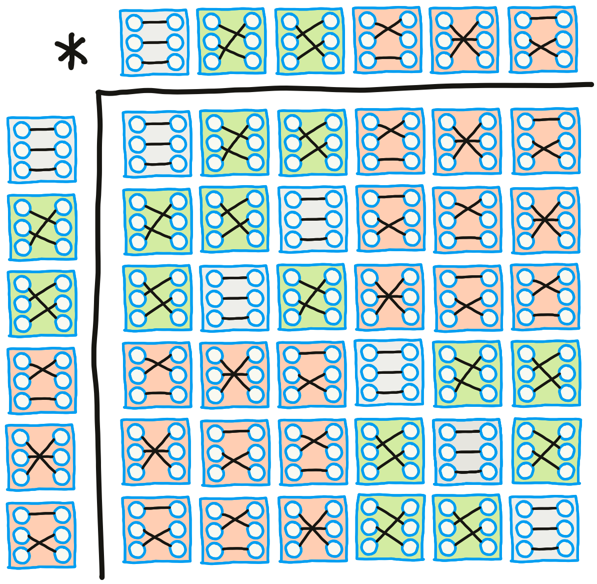
Identity

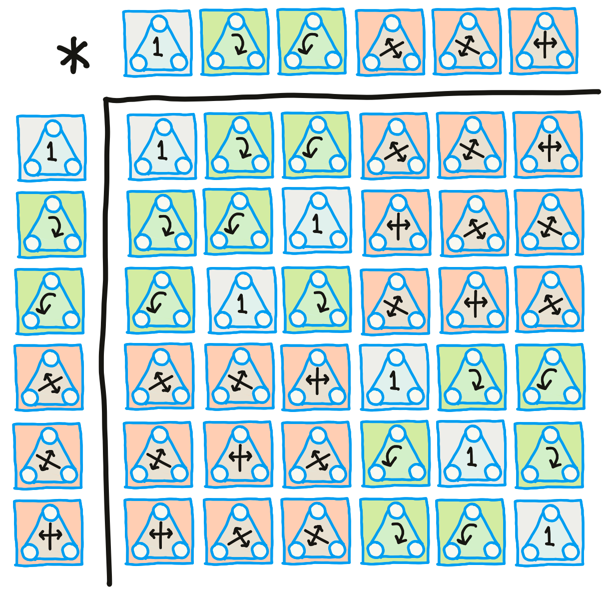


Inverses

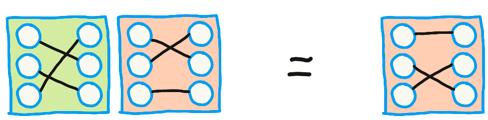


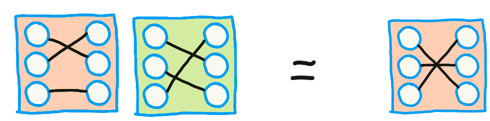
S_3 multiplication tables





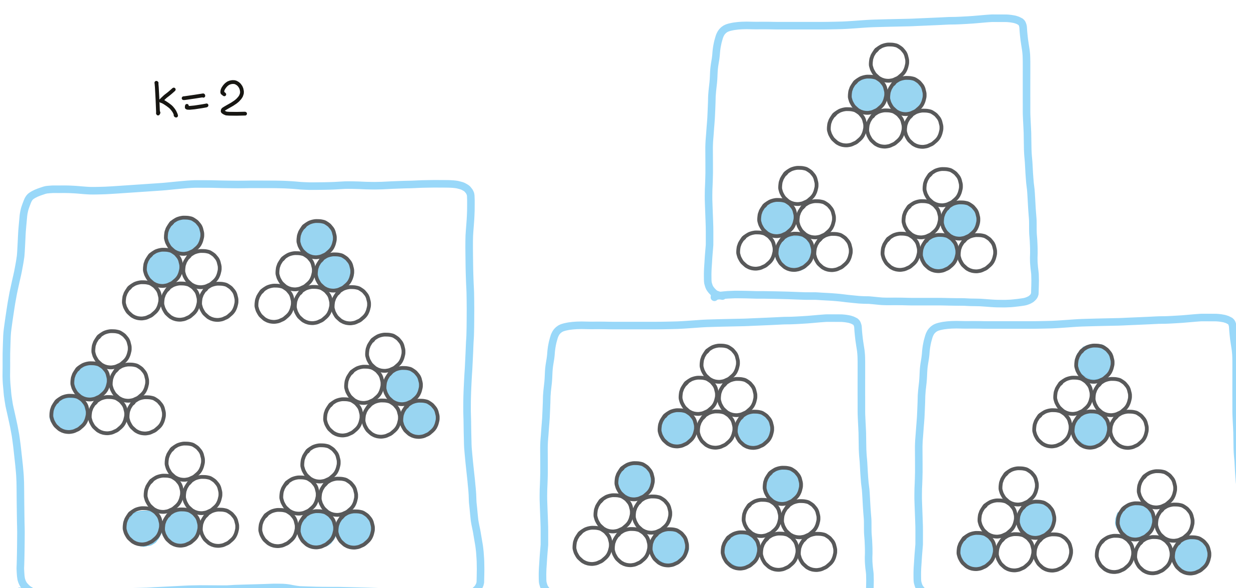
Not commutative



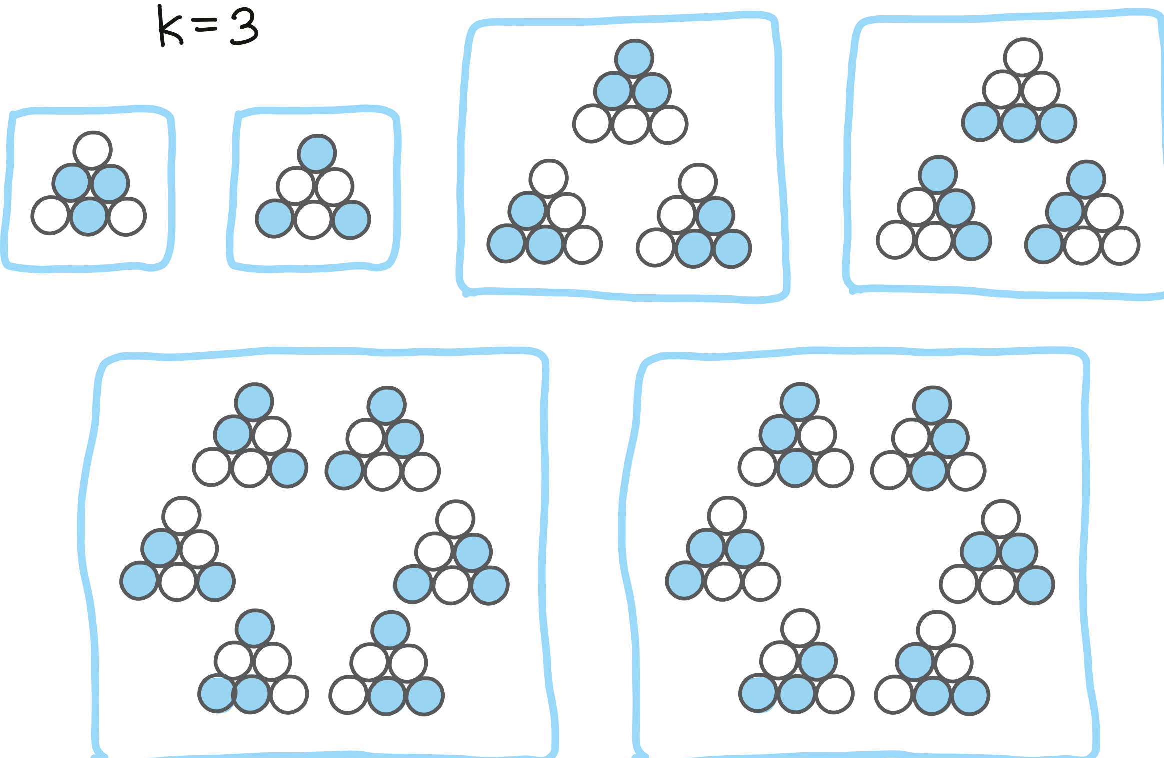


Counting problem: Mark k cells in a triangular grid
How many patterns, up to S_3 symmetry?

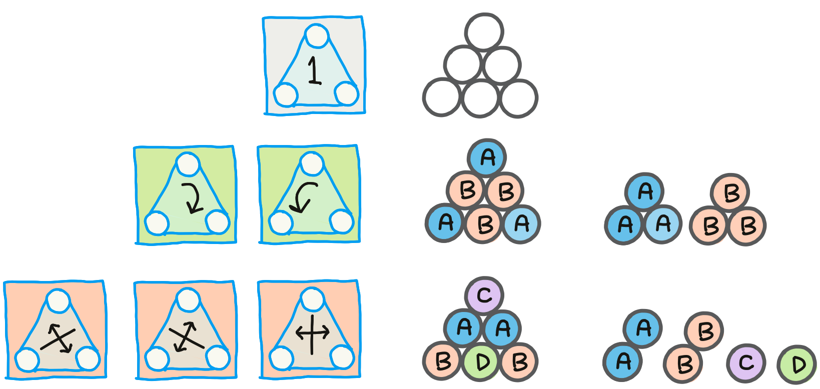
$k=2$



$k=3$



$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g|$$

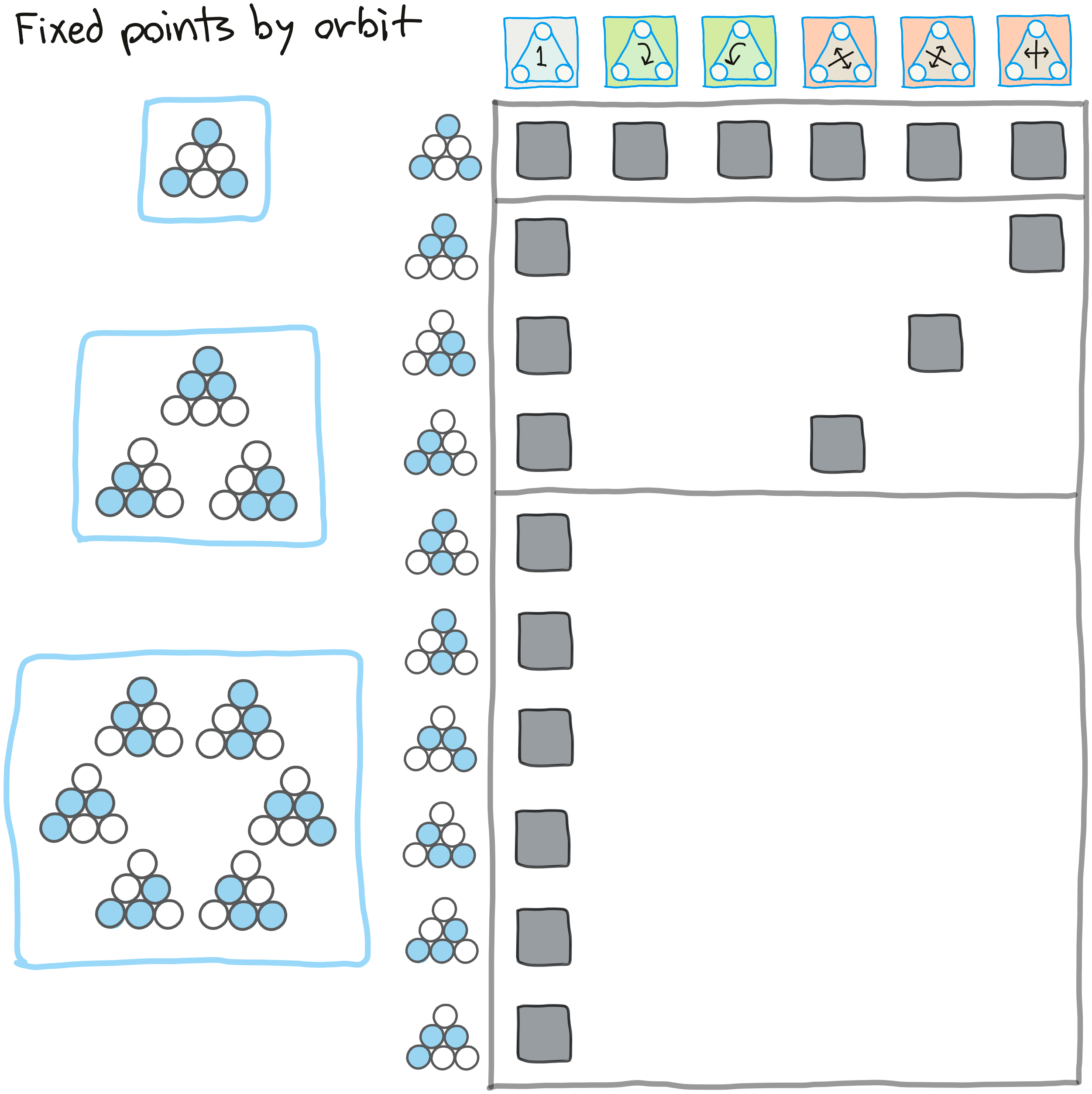


$k=2$	$k=3$
$\binom{6}{2}$	$\binom{6}{3}$
0	$\binom{2}{1}$
$\binom{2}{1}+\binom{2}{2}$	$\binom{2}{1}\binom{2}{1}$

$k=2$: $\frac{1}{6}\left[\binom{6}{2}+3\left(\binom{2}{1}+\binom{2}{2}\right)\right] = \frac{1}{6}(15+3\cdot 3) = 4$ ☑

$k=3$: $\frac{1}{6}\left[\binom{6}{3}+2\binom{2}{1}+3\binom{2}{1}\binom{2}{1}\right] = \frac{1}{6}(20+2\cdot 2+3\cdot 4) = 6$ ☑

Fixed points by orbit

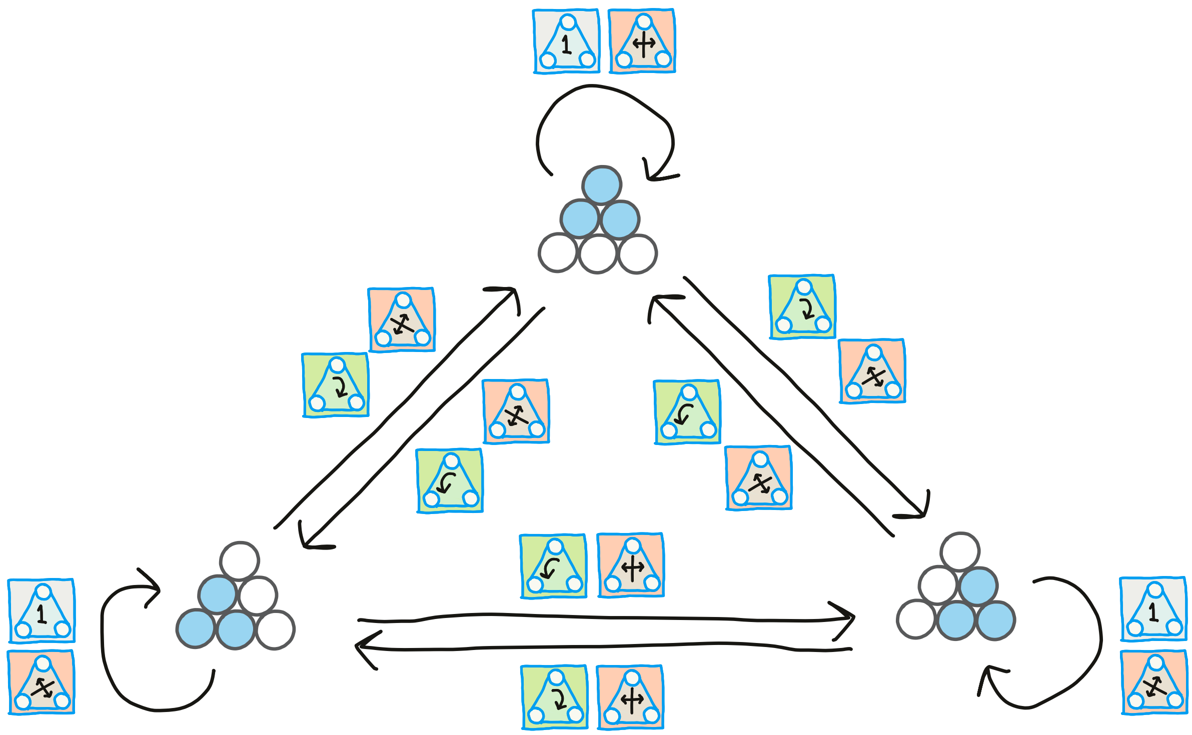


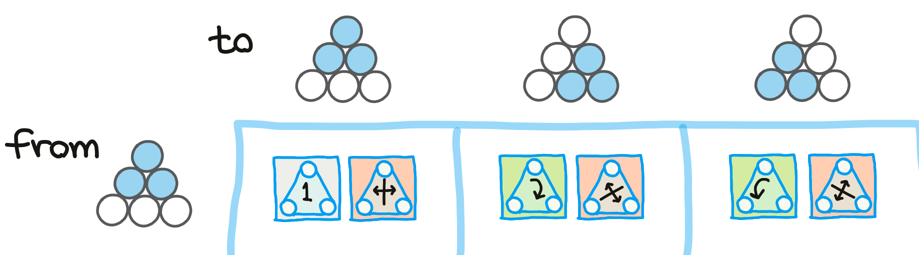
$\sum_{g \in G} |X_g|$ counts all fixed points (g,x) where $gx=x$

If we can understand why there are $|G|$ fixed points per orbit,

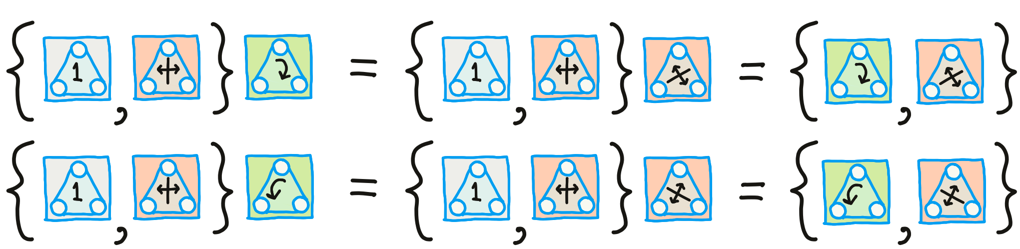
then we understand $|P| = \frac{1}{|G|} \sum_{g \in G} |X_g|$

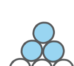
Look closely at how G acts on a particular orbit

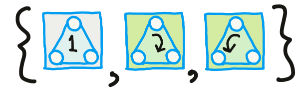


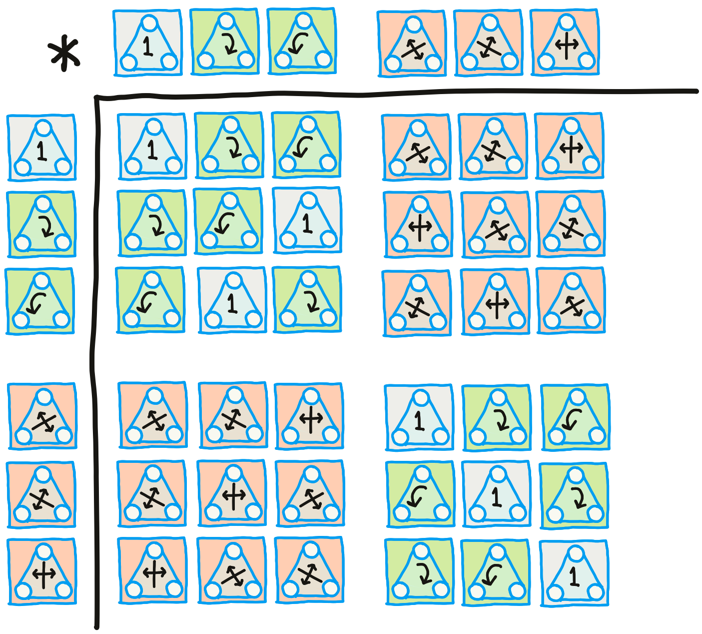


These subsets of G (cosets) are always in 1:1 correspondence with each other, so they divide G into equal sized subsets.



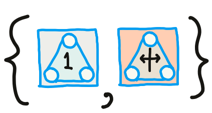
(# Fixed points of )(size of orbit) = |G|

Quotients: mod out by "normal subgroup" 

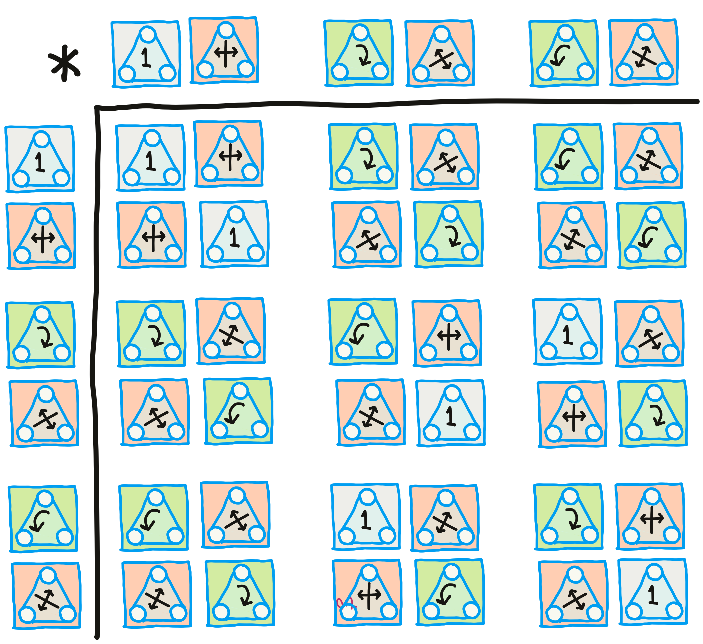


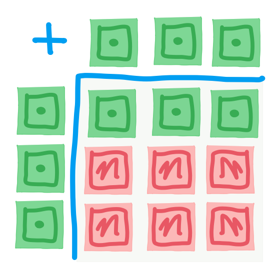
+	0	1
0	0	1
1	1	0

mod 2

 is not normal

and we don't get a coherent table when we try to mod out.



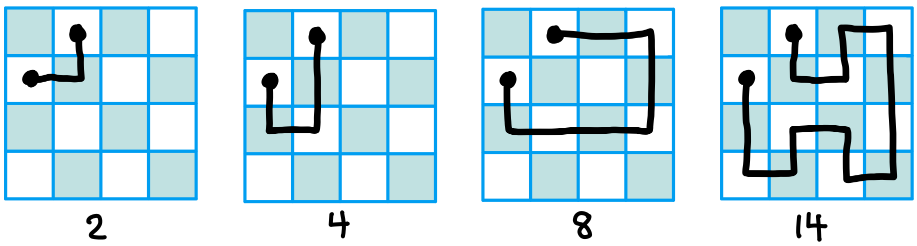


Various entries are inconsistent

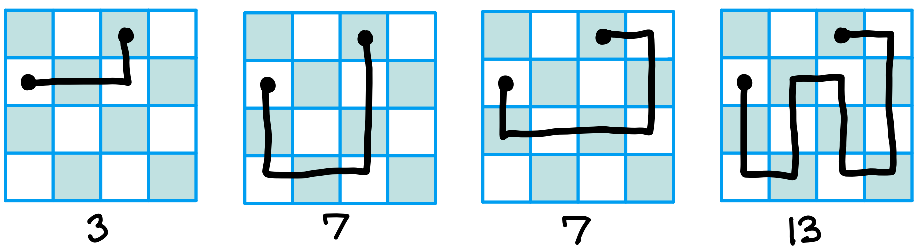
Expand on class questions:

Even-odd parity.

Walks alternate square colors



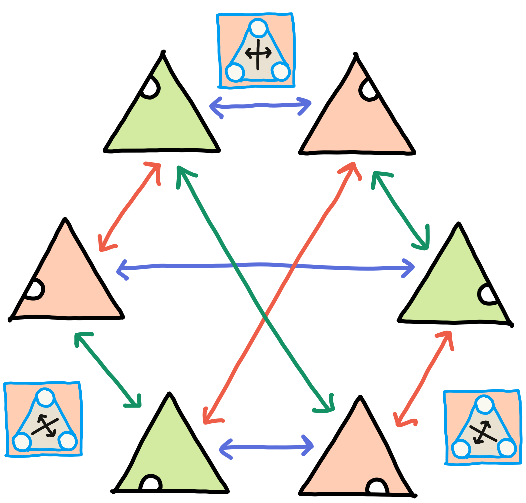
Walks between squares of the same color: even # steps

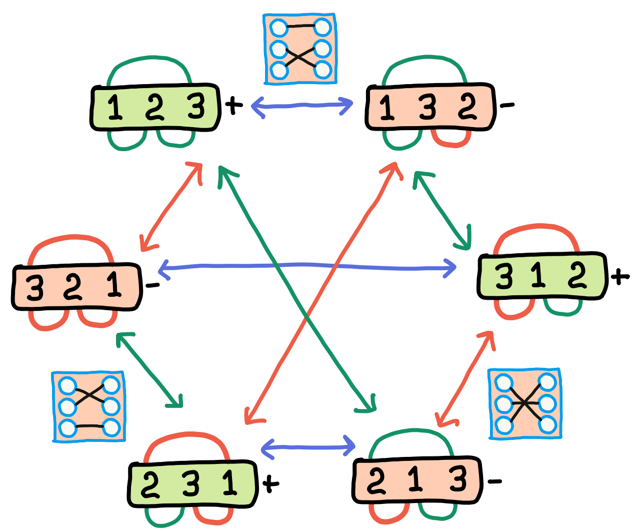


Walks between squares of the opposite color: odd # steps

We can checkerboard the graph of all triangle positions.

Flips all change checkerboard color

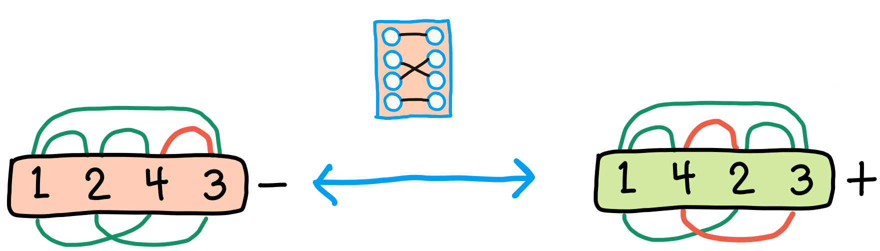




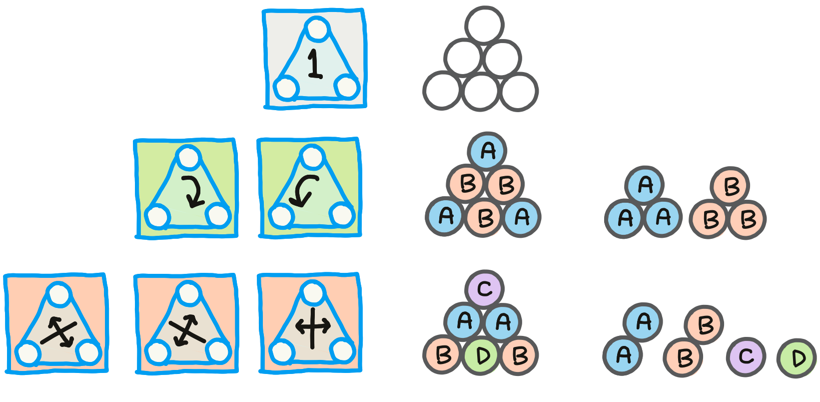
We can checkerboard the graph of all permutations of $\{1, .., n\}$

Even-odd: How many pairs are out of order?

Adjacent pair swaps change this count by 1



k colors $\quad |P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$

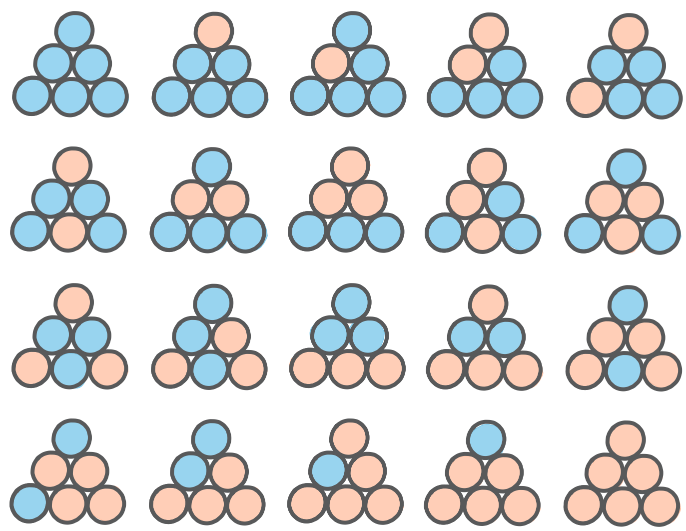


k^6

k^2

k^4

$k=2$



$$|P| = \frac{1}{6}(k^6+2k^2+3k^4)$$

$$= \frac{1}{6}(\underbrace{64+2\cdot 4}_{12}+\underbrace{3\cdot 16}_{8})$$

$$= 20$$

$k=3 \quad |P| = \frac{1}{6}(k^6+2k^2+3k^4) = \frac{1}{6}(729+2\cdot 9+3\cdot 81) = 165$

use 1 color: 3

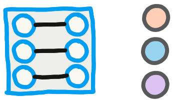
use 2 colors: $\binom{3}{2}18$ (From above)

$\Rightarrow$ use 3 colors: $165-3-\binom{3}{2}18 = 108$

Not easily checked

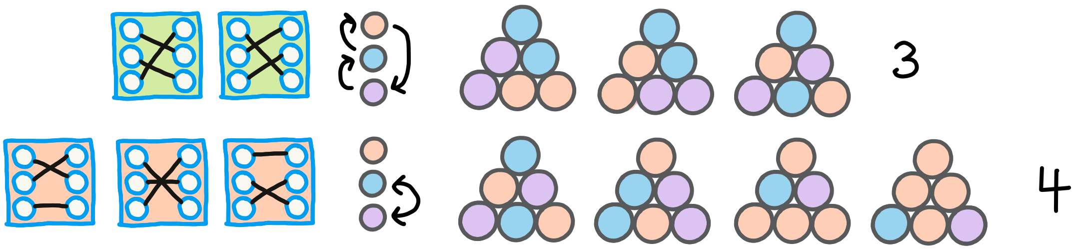
(This way lies madness)

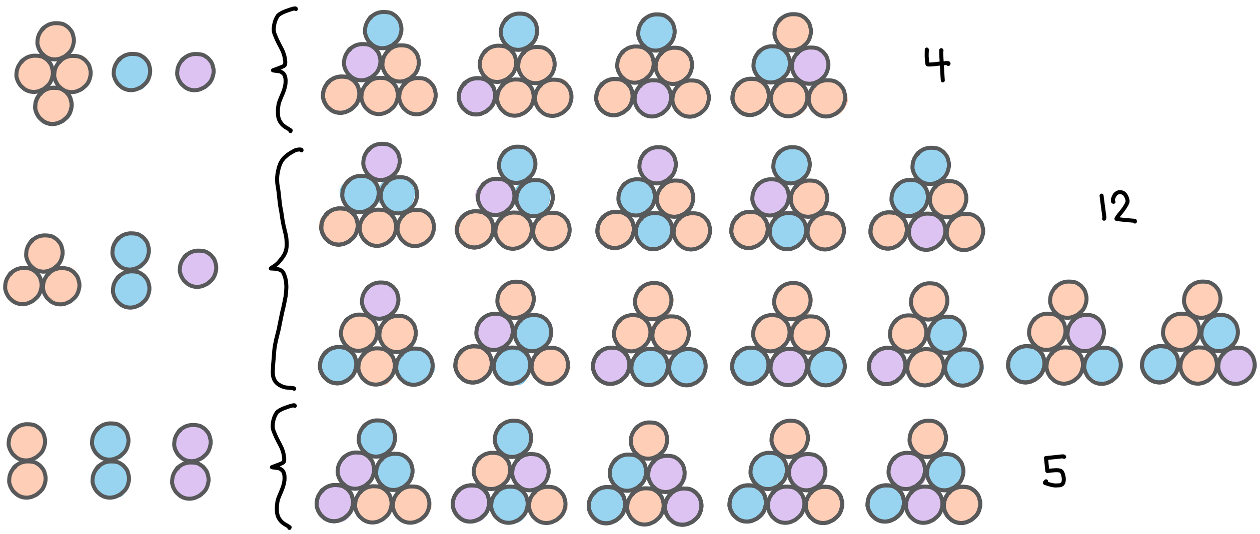
Let S_3 act on the colors, for this $|X| = 108$



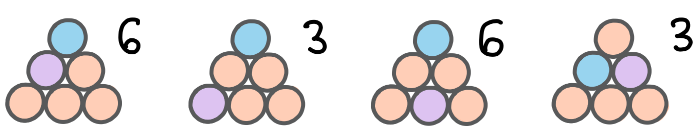
108

$$\frac{1}{6}\Big(\underset{18}{108} + \underset{1}{2\cdot 3} + \underset{2}{3\cdot 4}\Big) = 21$$

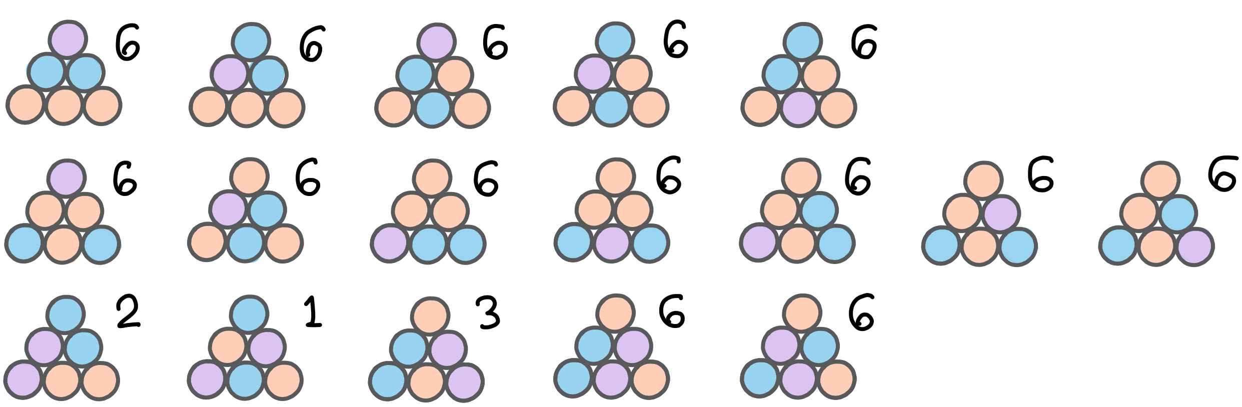




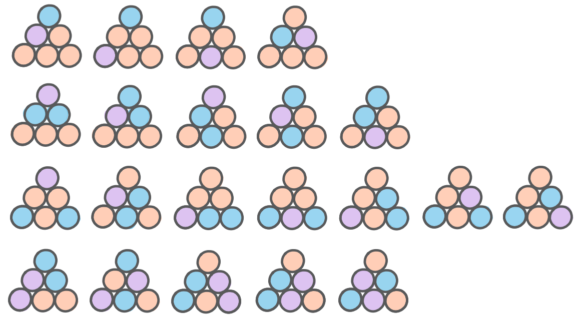
Now count orbit sizes by S_3 acting on colors

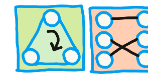


$$\boxed{\underset{96}{16\cdot 6} + \underset{9}{3\cdot 3} + \underset{2}{2\cdot 1} + \underset{1}{1\cdot 1} = 108}$$

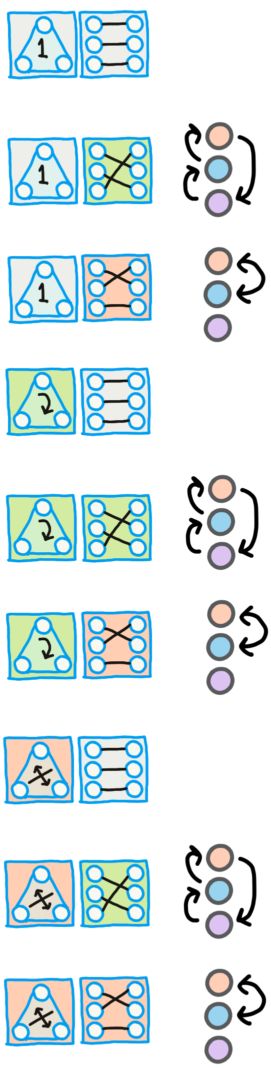


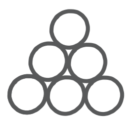
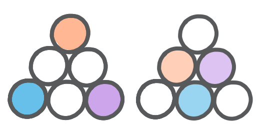
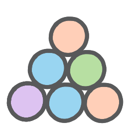
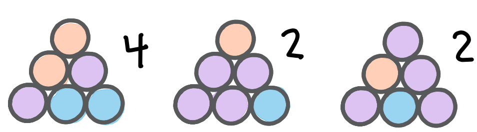
More systematic way to get 21 ways to color  using 3 interchangeable colors up to triangle symmetries:



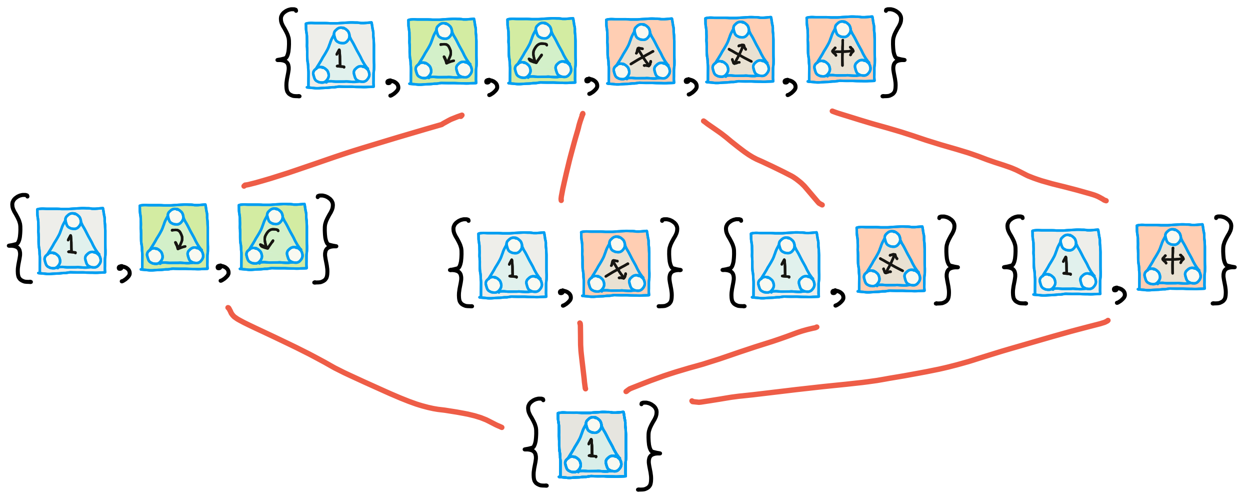
Let $G = S_3 \times S_3$, group of pairs of actions of form  acting on triangle and then color choices

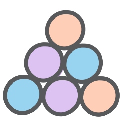
$|G| = |S_3||S_3| = 6 \cdot 6 = 36$



15	1		$3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$
	2	none	$\frac{1}{36}(540 + 4 \cdot 9 + 3 \cdot 36 + 9 \cdot 8) = 21$ (15, 1, 3, 2)
	3	none	
	2	none	
1	4		$3 \cdot 3 = 9$
	6	none	
3	3		4 zones color using all 3 colors $3^4 - 3 \cdot 2^4 + 3 = 81 - 48 + 3 = 36$
	6	none	
2	9		8
21	36		

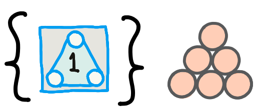
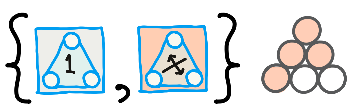
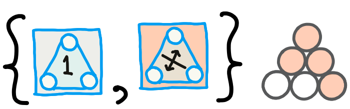
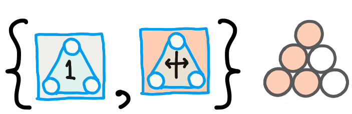
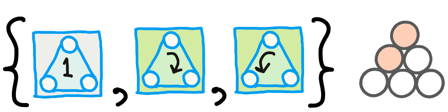
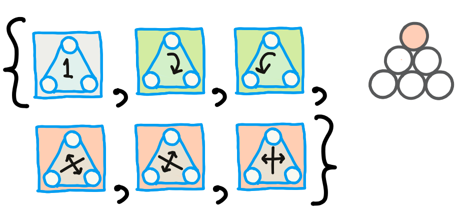
Can we use inclusion-exclusion instead of Burnside's lemma? Need to consider poset of subgroups of S_3. Möbius inversion.





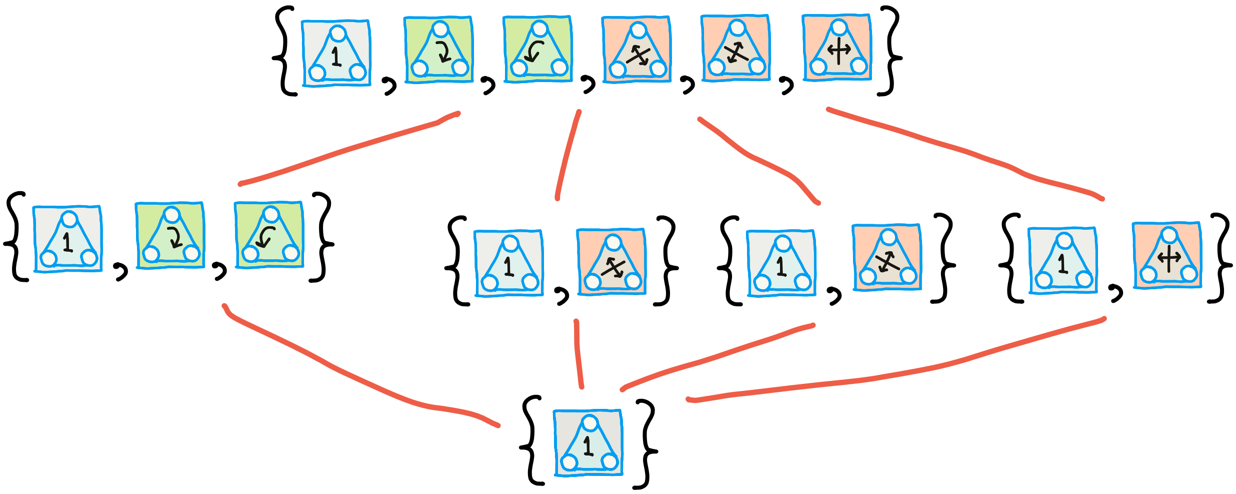
k colors

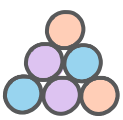
$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$$

type of symmetry	at least	exactly, divided by symmetries	(k^6	k^4	k^2	k)/6
	k^6	$\frac{1}{6}(k^6-3k^4-k^2+3k)$	1	-3	-1	3
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^2	$\frac{1}{2}(k^2-k)$			3	-3
	k	k				6
			1	3	2	0

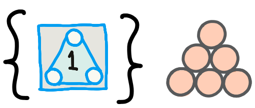
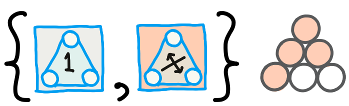
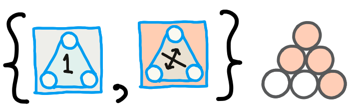
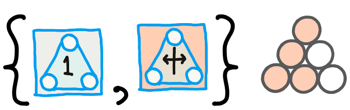
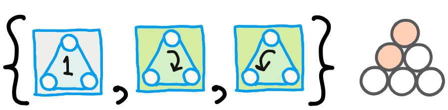
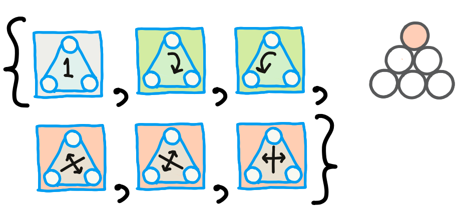
$$\frac{1}{6}(k^6+2k^2+3k^4)$$ ☑

Better approach: Skip Möbius inversion to compute "exactly".
Rather, when a pattern has d versions, we want to count each one with weight 1/d.
Work up the poset, adjusting weights based on count so far from below.





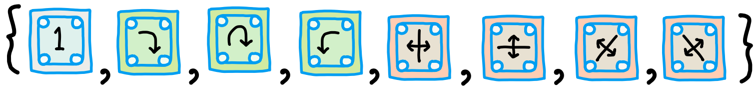
k colors $\qquad |P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$

type of symmetry	at least	desired weight	subtract below	net contribution
	k^6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}k^6$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}k^2$
	k	1	0	

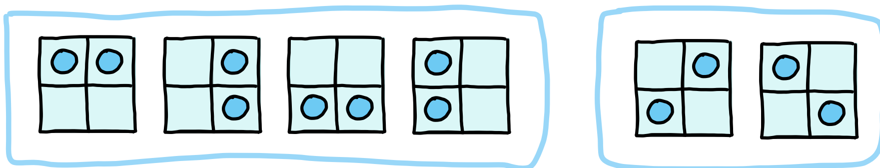
$\frac{1}{6}(k^6+2k^2+3k^4)$ ☑

This can be easier than Burnside's lemma.

Placing k markers on an n×n board, up to symmetry.

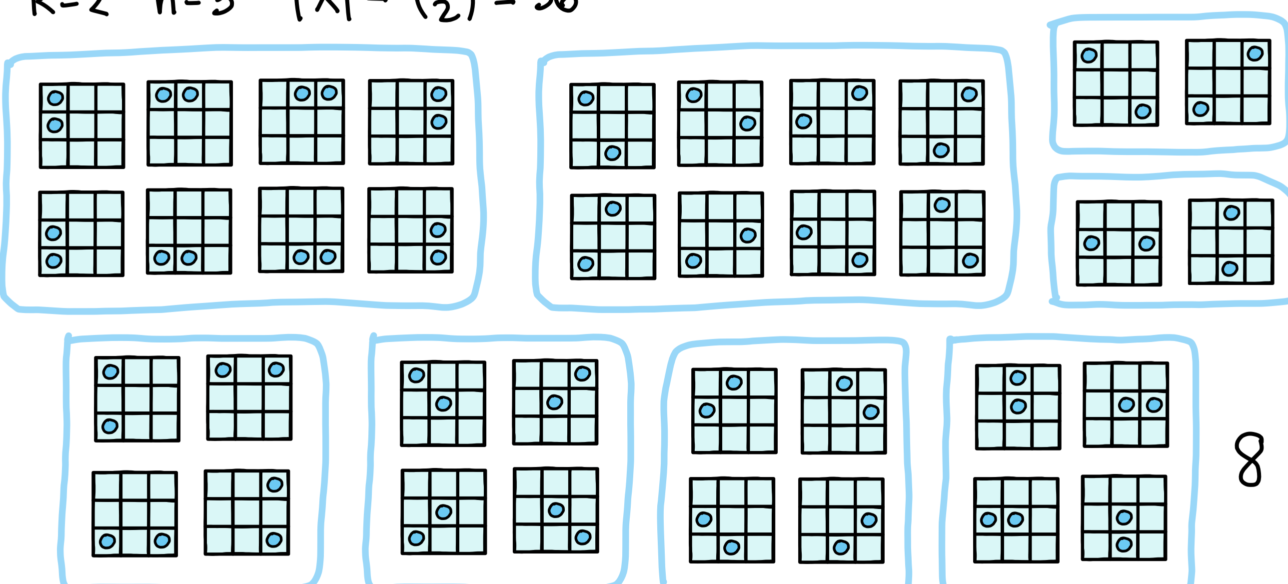
$G = \{$  $\}$ $\quad |G| = 8$

$k=n=2 \quad |X| = \binom{4}{2} = 6$

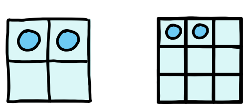


2

$K=2 \quad n=3 \quad |X| = \binom{9}{2} = 36$

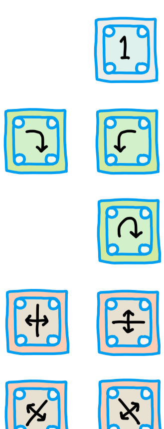


8



$\frac{1}{8}(6+2+2\cdot2+2\cdot2) = 2$ ☑

$\frac{1}{8}(36+4+2\cdot6+2\cdot6) = 8$ ☑



6	36
0	0
2	4
2	6
2	6