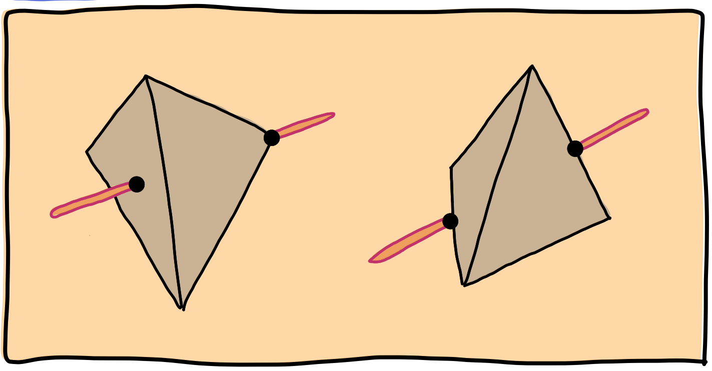
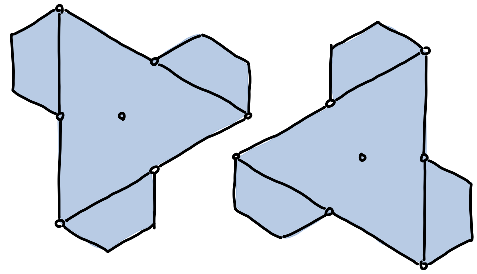
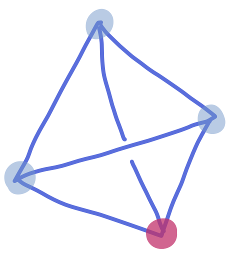
## March 11



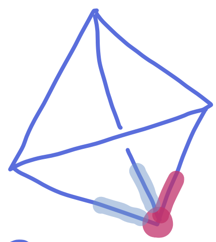


I have posted plans for the above model on our website.

The tetrahedron has 12 symmetries:



① Choose a corner
4 choices



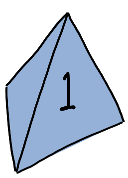
② choose an edge meeting that corner
3 choices

$G$ = group of symmetries of tetrahedron in $\mathbb{R}^3$
(we ignore flips through $\mathbb{R}^4$)

$|G| = 4 \cdot 3 \quad 12$

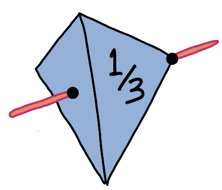
Can we find these 12 symmetries?

1



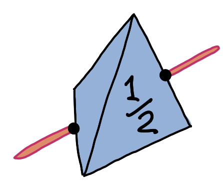

Identity
Do nothing

$2 \cdot 4$




1/3 turn either way
axis through face and vertex

3



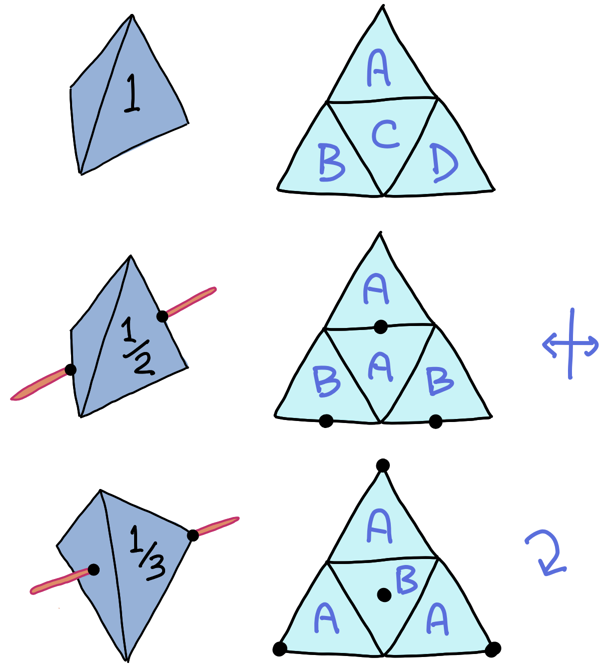

1/2 turn
axis through opposite edges

$1 + 2 \cdot 4 + 3 = 12$ ☑

**Burnside's lemma:**

$$\frac{1}{|G|} \sum_{g \in G} |X_g|$$

Example: How many ways can we color the sides of a tetrahedron, up to symmetry, using $k$ colors?




1 — $k^4$

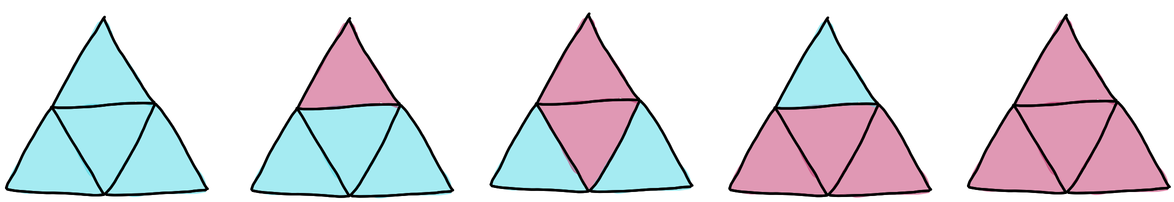
3 — $k^2$

8 — $k^2$

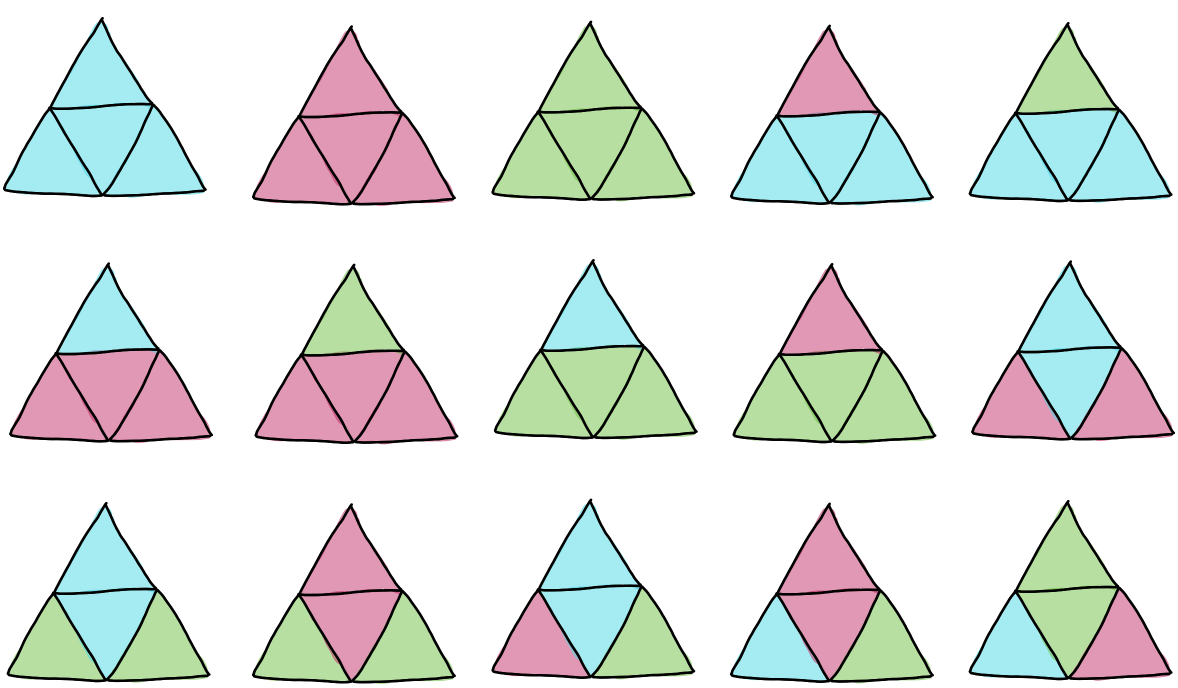
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(k^4+11k^2\right)}$$

| k | # | |
|---|---|---|
| 1 | 1 | |
| 2 | 5 | 16+44 |
| 3 | 15 | 81+99 |
| 4 | 36 | 256+176 |

Check: k=2 5 ☑

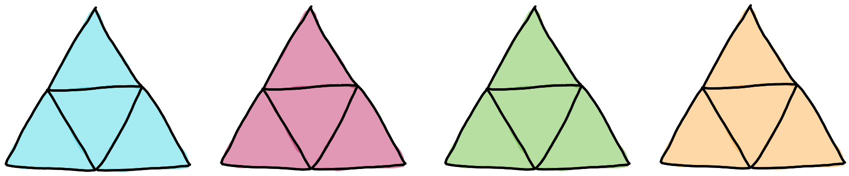


Check: k=3 15 ☑

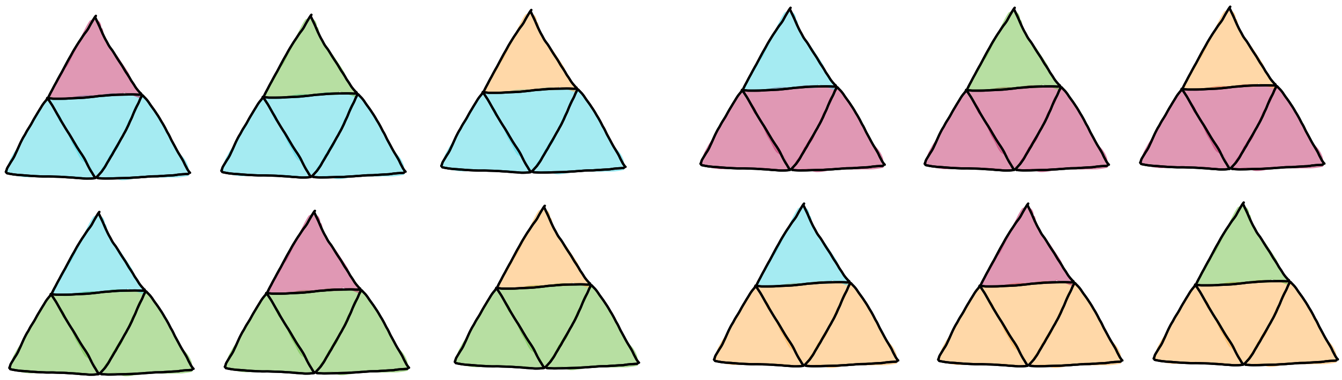


Check: k = 4  36 ☑

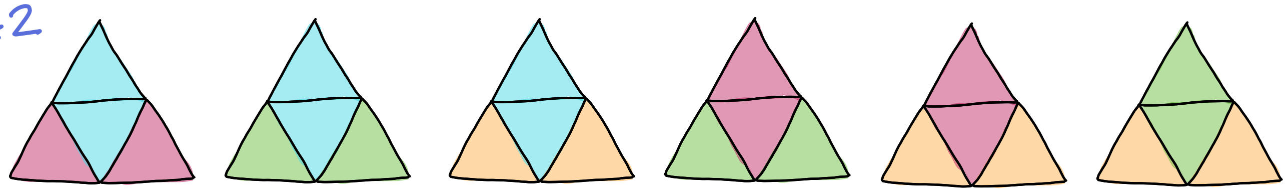
4



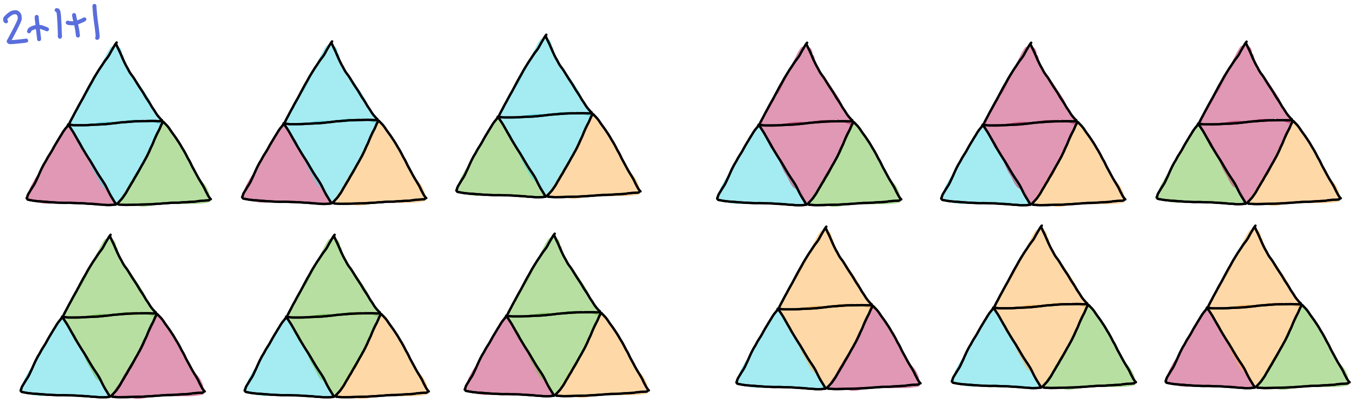
3+1

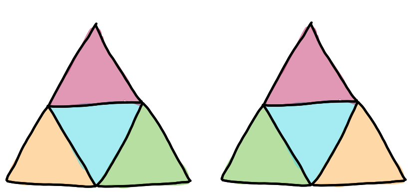


2+2



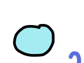
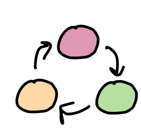
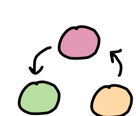
2+1+1





1+1+1+1

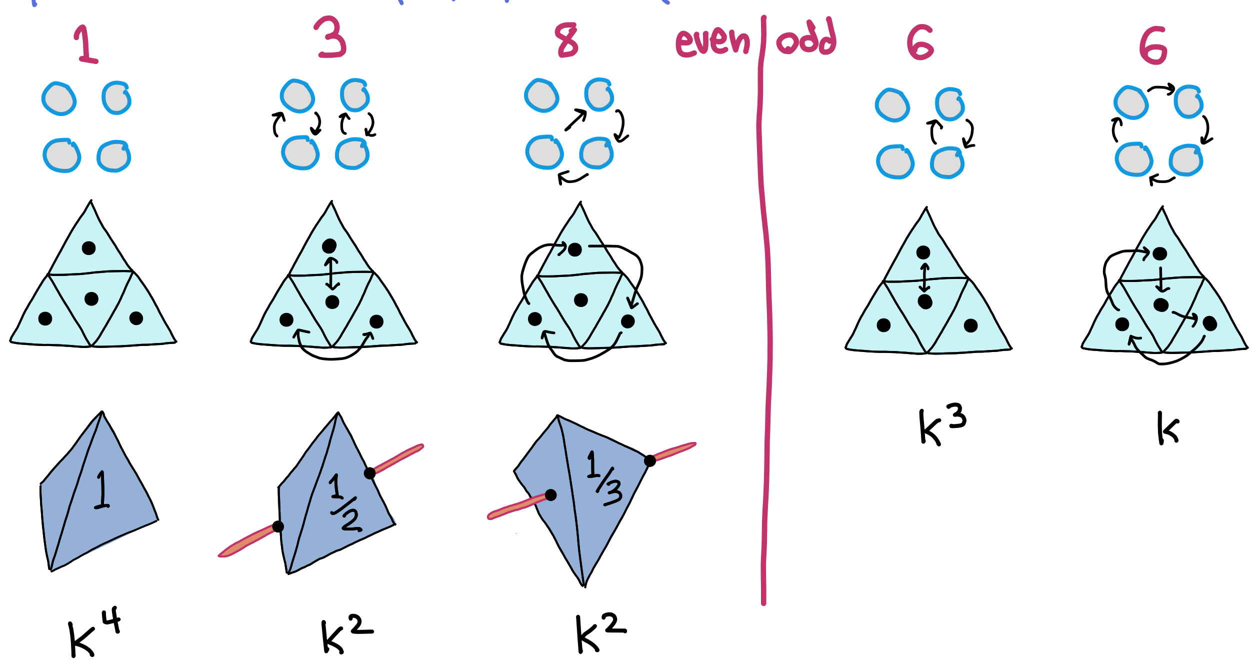
Finally a chiral pair

Look at , see  or 

This tells us that if we allow flips, we'll get

| $k$ | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| $G$ | 1 | 5 | 15 | 36 | (no flips) |
| $S_4$ | 1 | 5 | 15 | 35 | (flips in $\mathbb{R}^4$) |

$|S_4| = 4! = 24$ breaks up by cycle decomposition




$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}\left(k^4 + 6k^3 + 11k^2 + 6k\right)$$

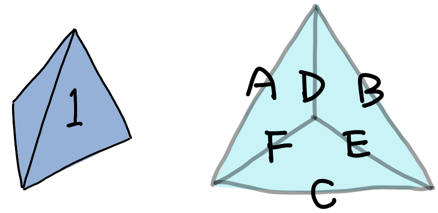
| $k$ | $k^2$ | $k^3$ | $k^4$ | $6k$ | $11k^2$ | $6k^3$ | $k^4$ | $\Sigma$ | # | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 6 | 11 | 6 | 1 | 24 | 1 | as before |
| 2 | 4 | 8 | 16 | 12 | 44 | 48 | 16 | 120 | 5 | as before |
| 3 | 9 | 27 | 81 | 18 | 99 | 162 | 81 | 360 | 15 | as before |
| 4 | 16 | 64 | 256 | 24 | 176 | 384 | 256 | 840 | 35 ☑ | not 36 |

Choosing subsets of faces is restricted version of 2-coloring $\Rightarrow$ no chirality

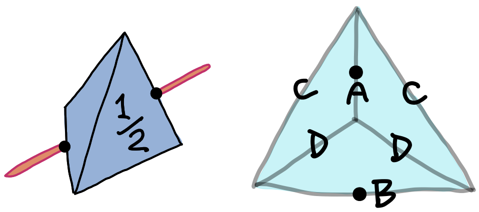
Coloring vertices is dual to coloring faces, same problem

- Coloring edges?
- Coloring everything?

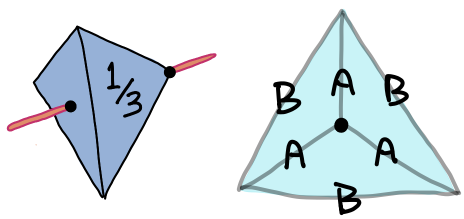
Coloring edges:

1 

$K^6$

3 

$K^4$

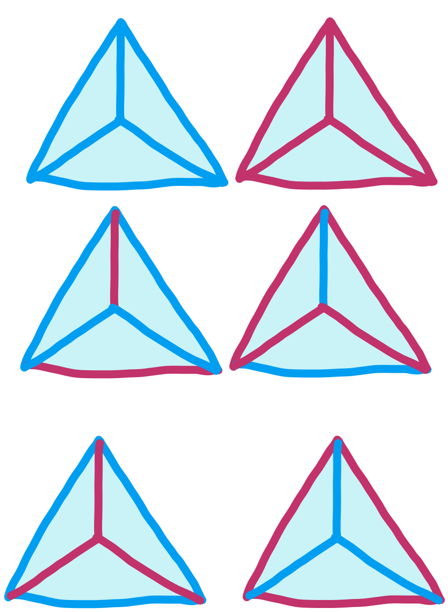
8 

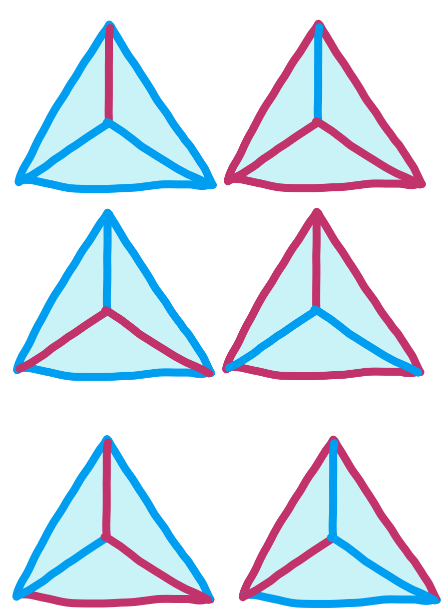
2 $K^2$

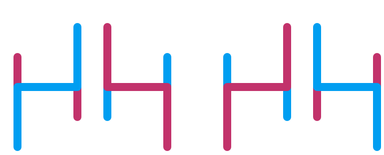
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(K^6+3K^4+8K^2\right)}$$

| K | 1 | 2 | 3 |
|---|---|---|---|
| $K^2$ | 1 | 4 | 9 |
| $K^4$ | 1 | 16 | 81 |
| $K^6$ | 1 | 64 | 729 |
| $8K^2$ | 8 | 32 | 72 |
| $3K^4$ | 3 | 48 | 243 |
| $K^6$ | 1 | 64 | 729 |
| $\Sigma$ | 12 | 144 | 1044 |
| # | 1 | 12 | 84 |

Check: K=2  12 ☑







(Corrected from class)