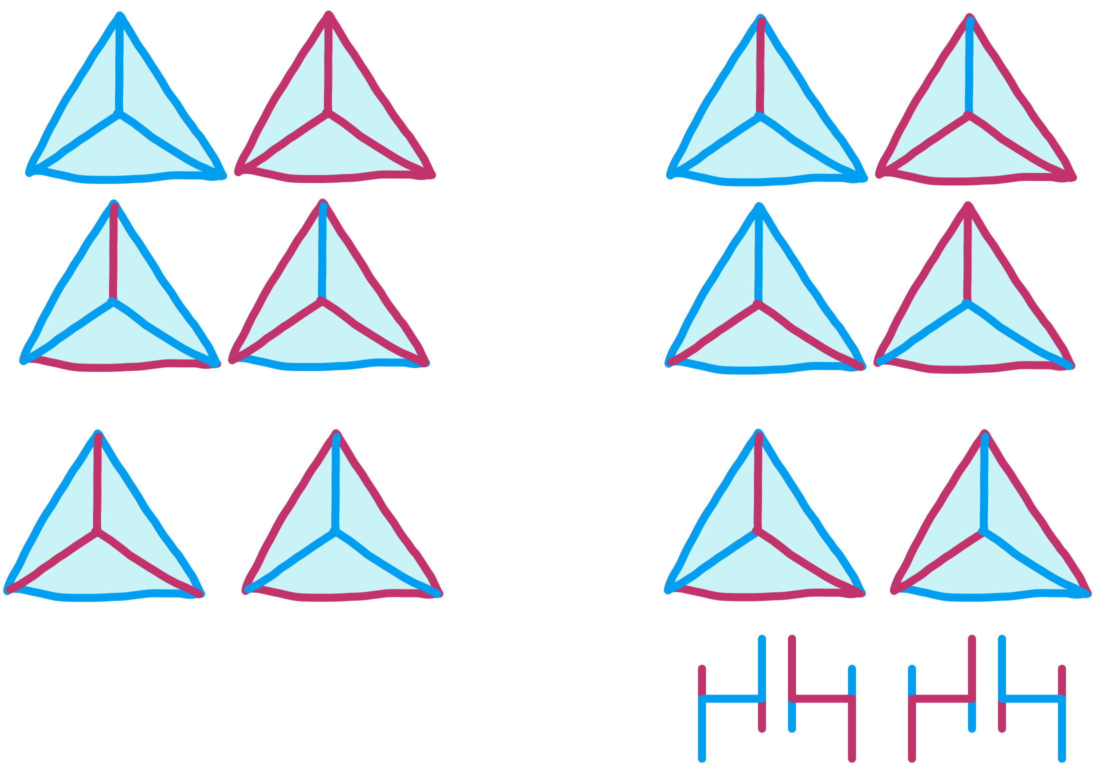
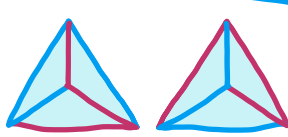
Tuesday, March 16

From last class: 12 ways to 2-color edges of a tetrahedron, up to symmetry

Check: $k=2$ 12

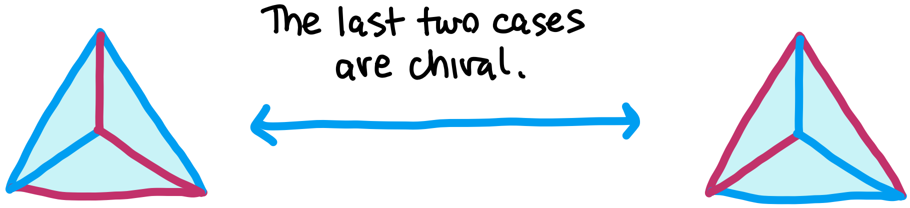


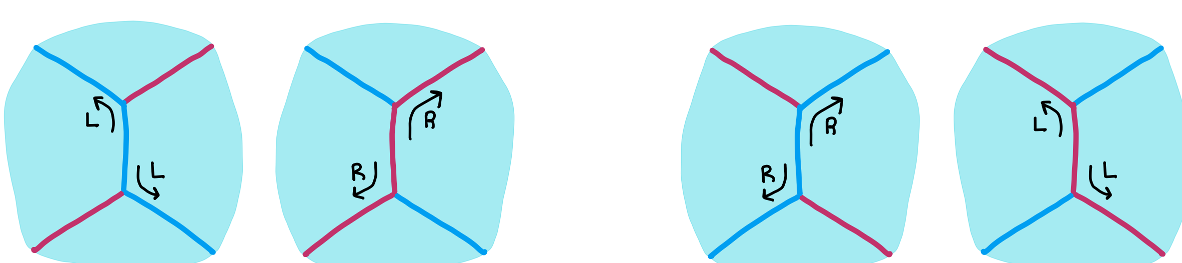
(Corrected from class)

In class I had: 

These were actually the same.

The last two cases are chiral.

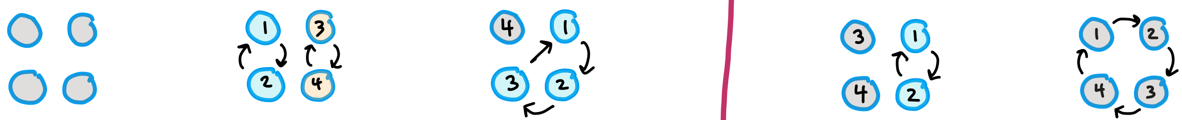




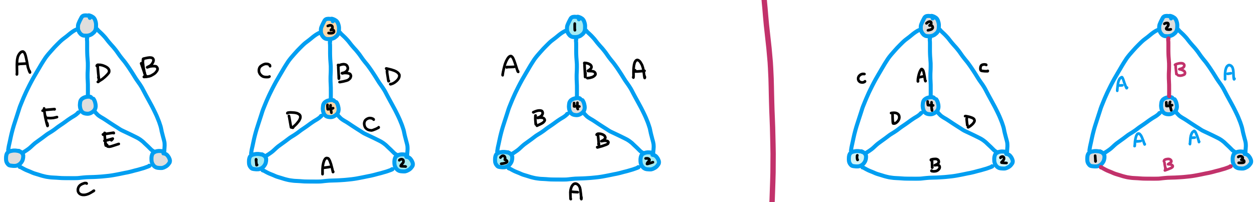
This tells us that including flips through $\mathbb{R}^4$, we should get 11 not 12

$|S_4| = 4! = 24$ breaks up by cycle decomposition

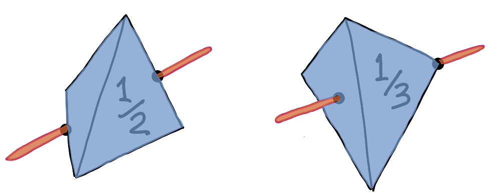
1	3	8	even | odd	6	6



	A (1,2) B (3,4) C (1,3) (2,4) D (1,4) (2,3)	A (1,2) (2,3) (1,3) B (1,4) (2,4) (3,4)	A (3,4) B (1,2) C (1,3) (2,3) D (1,4) (2,4)	A (1,2) (2,3) (3,4) (1,4) B (1,3) (2,4)



k^4 k^2

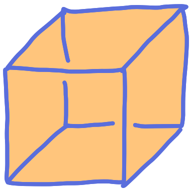


k^6 k^4 k^2

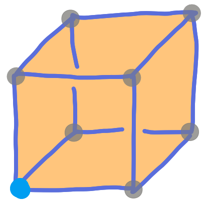
$$\frac{1}{|G|}\sum_{g \in G} |X_g| = \frac{1}{24}\left(k^6 + 9k^4 + 14k^2\right)$$

k	k^2	k^4	k^6	$14k^2$	$9k^4$	k^6	Σ	#
1	1	1	1	14	9	1	24	1
2	4	16	64	56	144	64	264	11 ☑

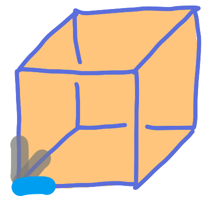
Symmetries of the cube



G = group of symmetries of cube in $\mathbb{R}^3$
(we ignore flips through $\mathbb{R}^4$)



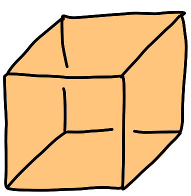
① Choose a corner
8 choices



② choose an edge meeting that corner
3 choices

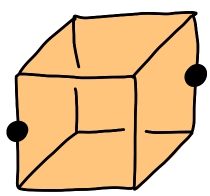
$|G| = 8 \cdot 3 = 24$

Can we find these 24 symmetries?



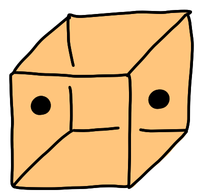
identity 1

1



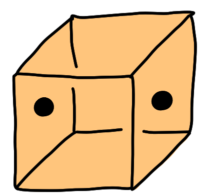
½ turn

6



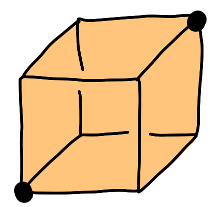
½ turn

3



¼ turn either way

6

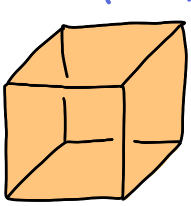


⅓ turn either way

8

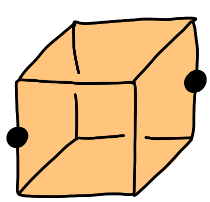
This $G \approx S_4$. Imagine 4 diagonal sticks inside the cube.
Easier: Label opposite corners the same, using $\{1,2,3,4\}$
Every permutation is possible.

How many ways can we k-color the faces of a cube, up to symmetry?



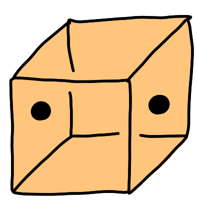
identity 1

k^6



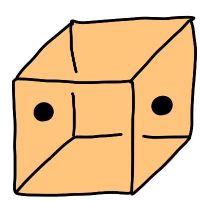
½ turn

$6k^3$



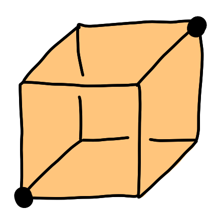
½ turn

$3k^4$



¼ turn either way

$6k^3$



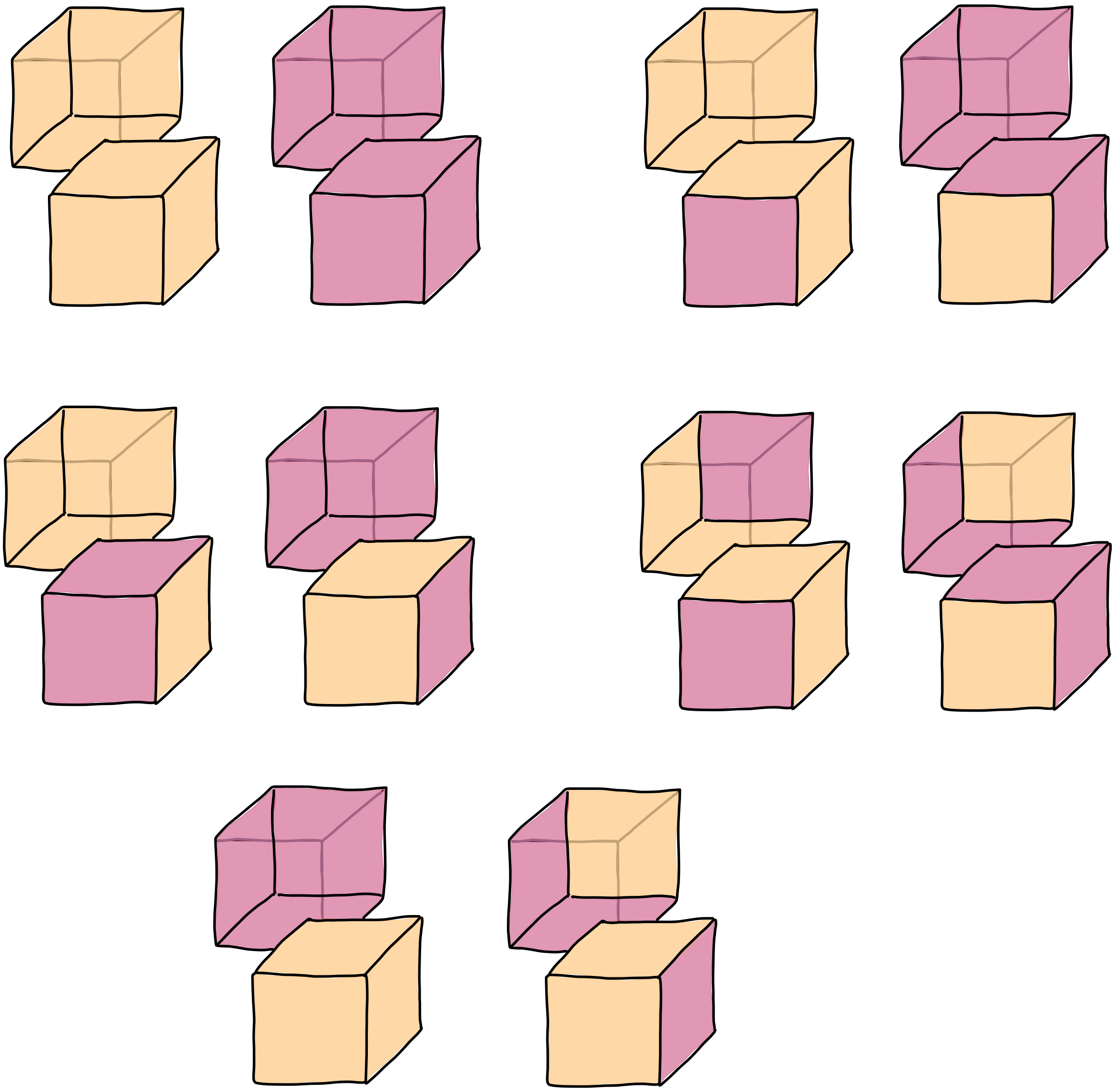
⅓ turn either way

$8k^2$

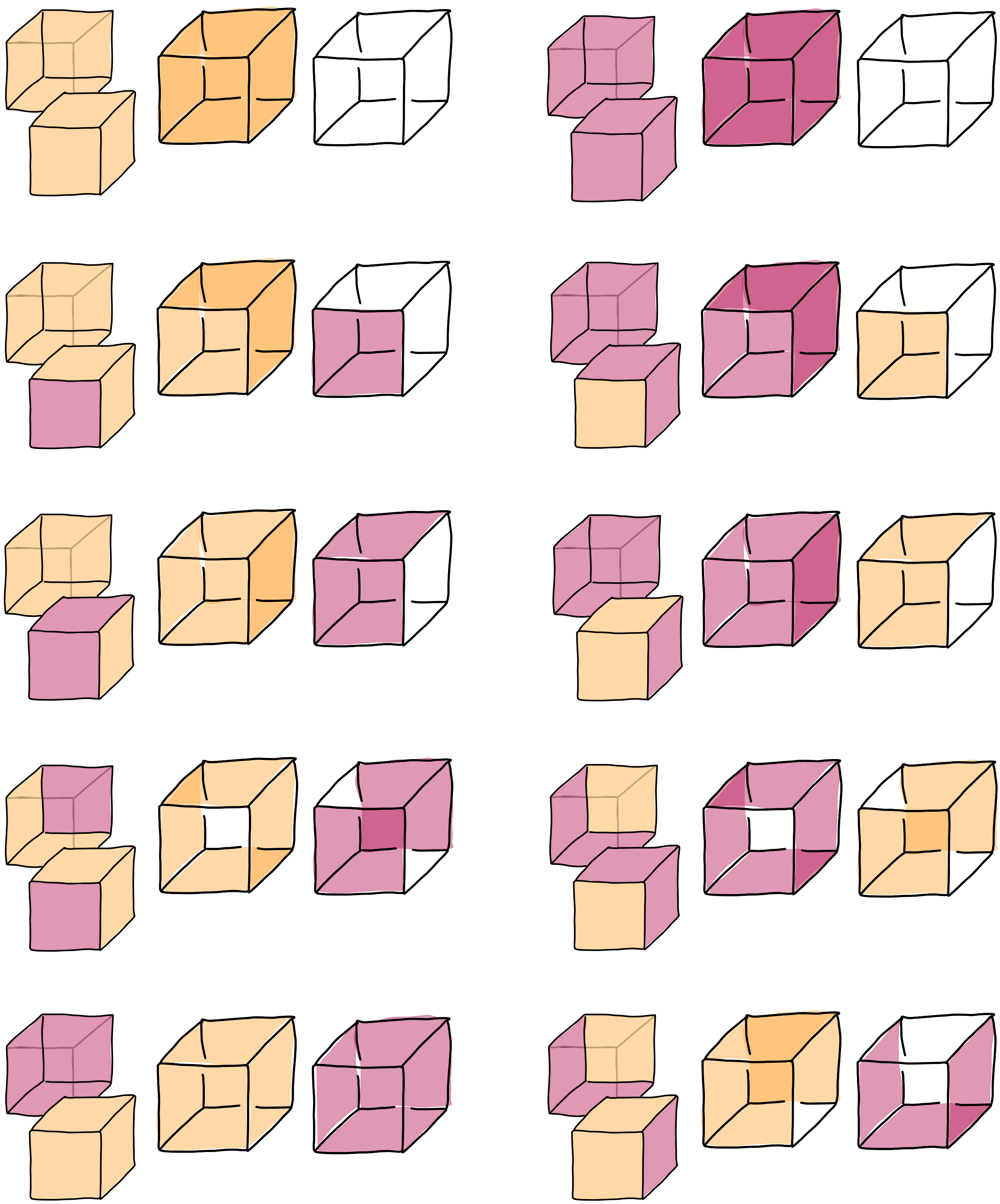
$$\frac{1}{24}\left(k^6 + 3k^4 + 12k^3 + 8k^2\right)$$

$$k=2 \Rightarrow \frac{1}{24}\left(64 + \underbrace{3\cdot 16}_{48} + \underbrace{12\cdot 8}_{96} + \underbrace{8\cdot 4}_{32}\right) = 240/24 = 10$$

Check k = 2 :   10



After class: Try another way to draw these.
Two wire frames per pattern, to separate the faces of each color.

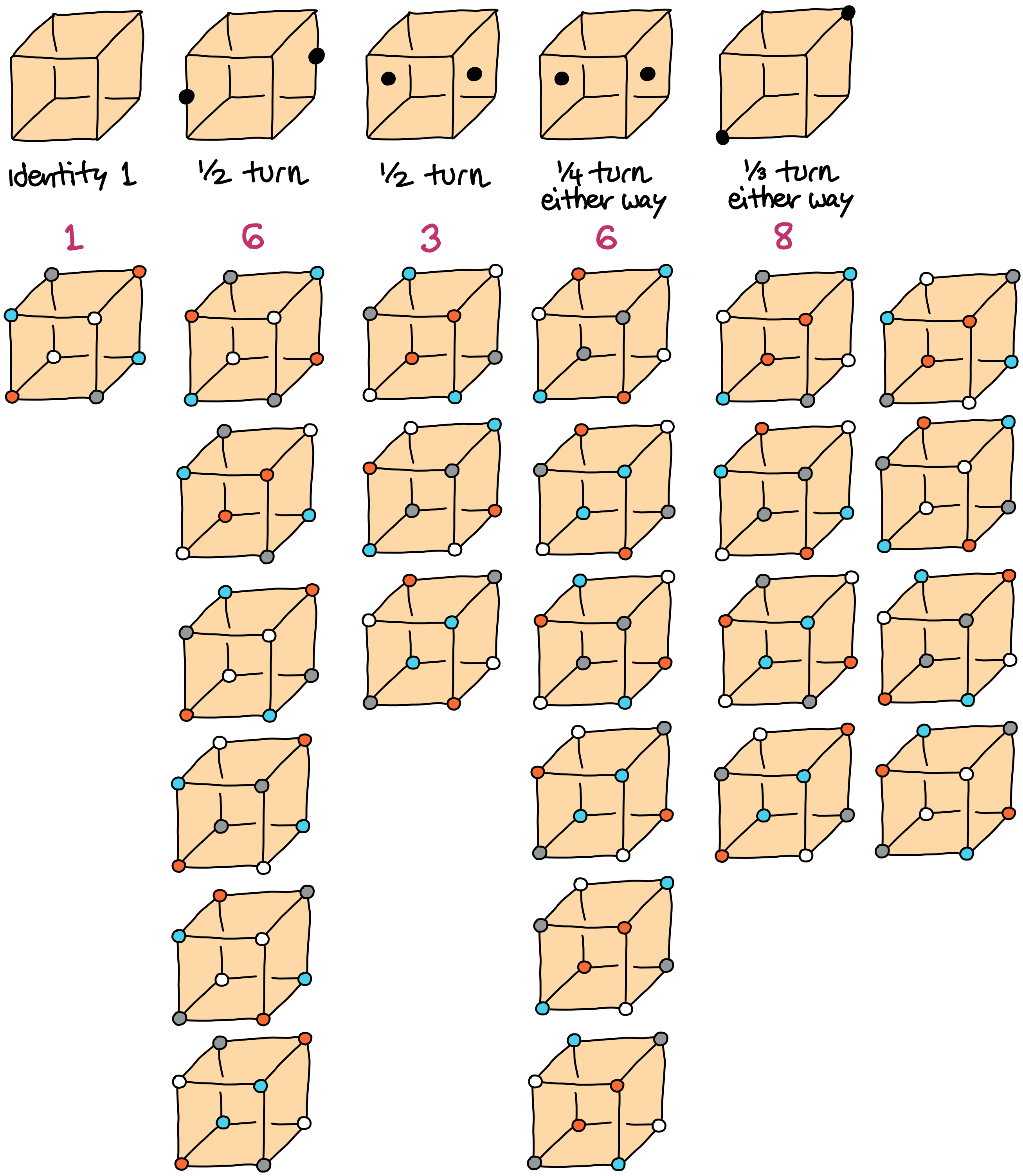


Check that action of S_4 induces every symmetry of cube

Four pairs of opposite corners, marked by ○ ● ● ●

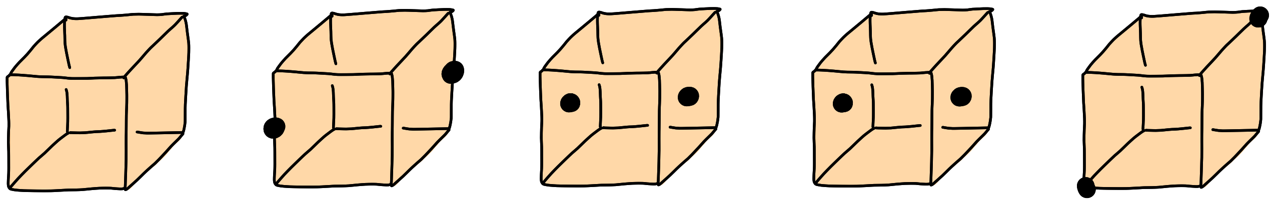
S_4 permutes these pairs

Every permutation corresponds to some rotation in space:

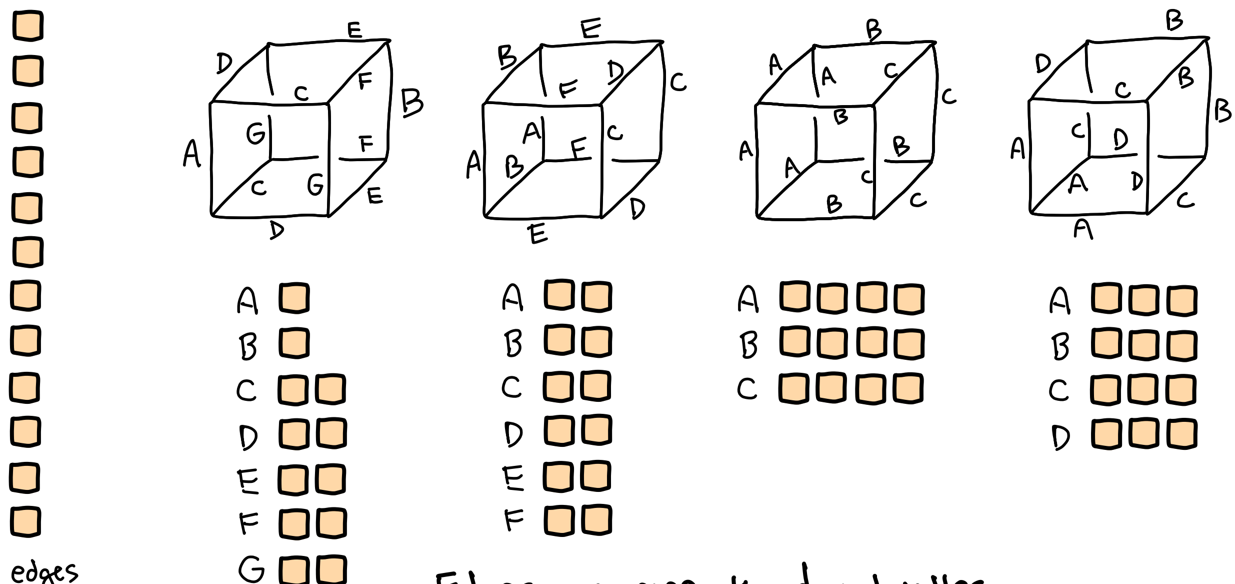


How many ways can we choose k edges of a cube, up to symmetry?

1	6	3	6	8



identity 1	½ turn	½ turn	¼ turn either way	⅓ turn either way

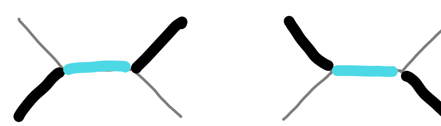


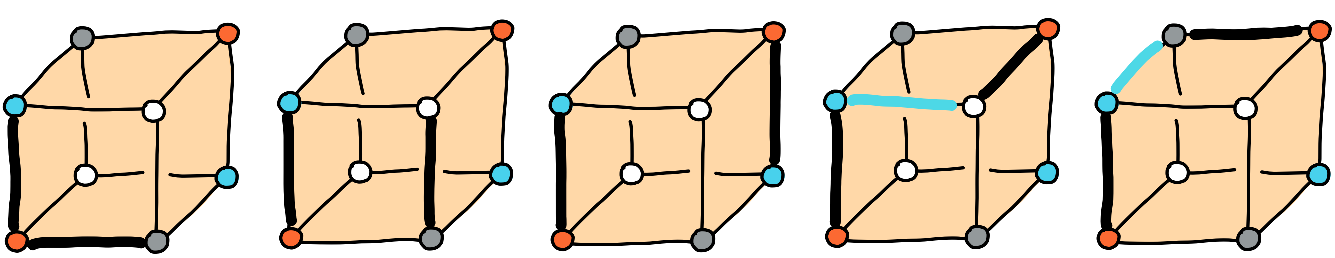
Edges come prepackaged in bundles
We need to make k buying entire bundles

K=2

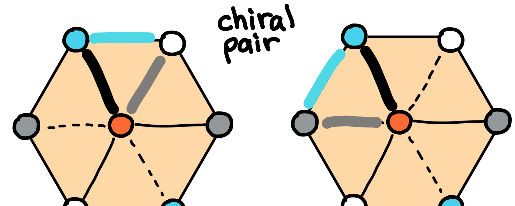
$\binom{12}{2}=66$	6	6	0	0

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}(66+6\cdot 6+3\cdot 6) = \frac{120}{24} = 5$$





only way to meet at a vertex	only way to use all four vertex colors	only way to use just two vertex colors



chiral pair